Lecture 30

Recap: Möbius transformations, also called linear fractional transformations (LFT) are conformal automorphisms of $\hat{\mathbb{C}}$

$$\mathrm{Aut}(\hat{\mathbb{C}}) = \{ M_A : A \in GL_2(\mathbb{C}) \}$$

for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $M_A(z) = \dfrac{az+b}{cz+d}$.

<u>Fundamental properties</u>: (1) $M_A \circ M_B = M_{AB}$; $M_{Id} = Id$,

i.e. $A \mapsto M_A$ is a group hom. $GL_2(\mathbb{C}) \to \mathrm{Aut}(\hat{\mathbb{C}})$

and gives an iso. $\underbrace{GL_2(\mathbb{C}) / \{\lambda \cdot Id : \lambda \in \mathbb{C} \setminus \{0\}\}}_{\text{also denoted by } PGL_2(\mathbb{C})} \xrightarrow{\sim} \mathrm{Aut}(\hat{\mathbb{C}})$

(2) Under a Möbius transformation, image of a circle or line is again a circle or line.

(3) $M \in \mathrm{Aut}(\hat{\mathbb{C}})$, $M \neq Id$ $\Rightarrow$ $|\{ z : M(z) = z \}| = 1$ or 2.

(4) Given $z_1, z_2, z_3 \in \hat{\mathbb{C}}$ three distinct points, $\exists !$ $M \in \mathrm{Aut}(\hat{\mathbb{C}})$ s.t. $M(z_1) = 0$, $M(z_2) = 1$, $M(z_3) = \infty$

($M(z) = \dfrac{z_2 - z_3}{z_2 - z_1} \dfrac{z - z_1}{z - z_3}$ if $\infty \notin \{z_1, z_2, z_3\}$

$= \dfrac{z_2 - z_3}{z - z_3}$ if $z_1 = \infty$

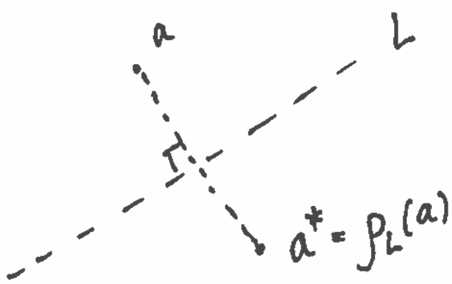
$$M(z) = \frac{z - z_1}{z - z_3}, \quad \text{if } z_2 = \infty$$

$$\frac{z - z_1}{z_2 - z_1}, \quad \text{if } z_3 = \infty. \;)$$

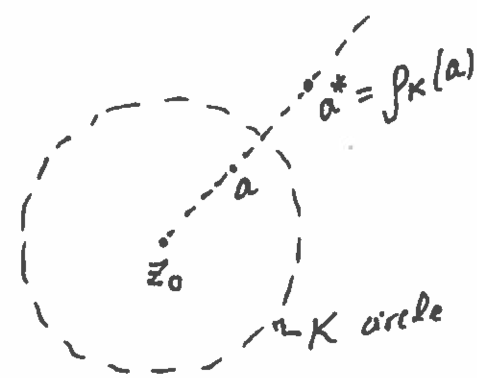
(5) $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$, 4 distinct points.

$[z_1 : z_2 : z_3 : z_4] := M(z_4)$ where $M \in \mathrm{Aut}(\hat{\mathbb{C}})$ is the unique Möbius transformation s.t. $M(z_1) = 0$, $M(z_2) = 1$, $M(z_3) = \infty$.

§1. Möbius tramformations preserve reflections in circles / lines



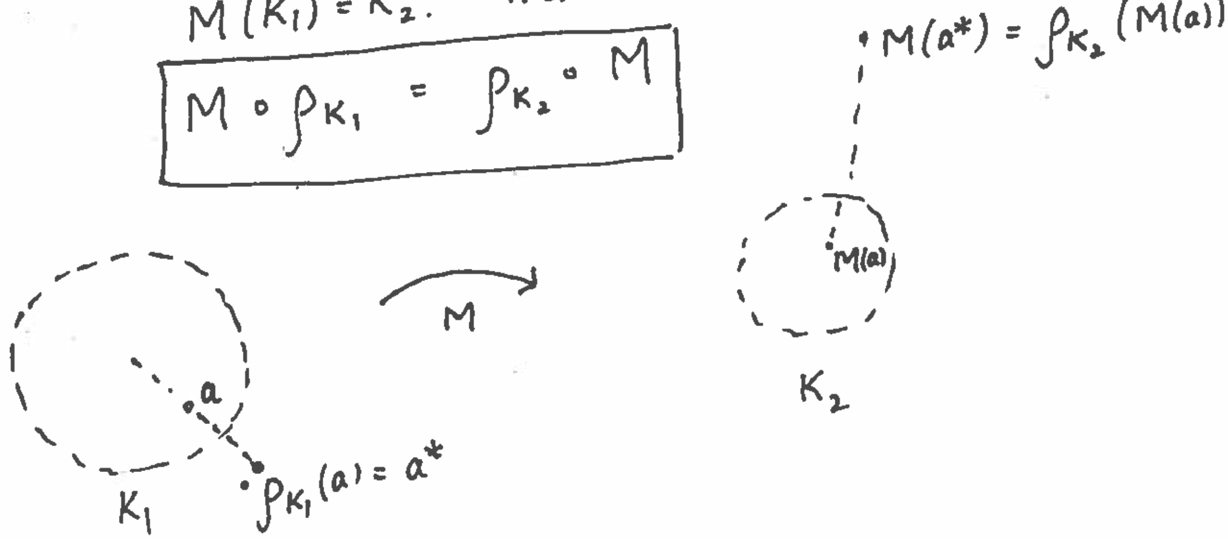
Reflection in a line



Reflection w.r.t. a circle

<u>Theorem.</u> – Let $M \in \mathrm{Aut}(\hat{\mathbb{C}})$; K_1, K_2 two circles (or lines) s.t. $M(K_1) = K_2$. Then

$$\boxed{M \circ \rho_{K_1} = \rho_{K_2} \circ M}$$



Formula for ρ_K : if $K = C(z_0; r)$ then

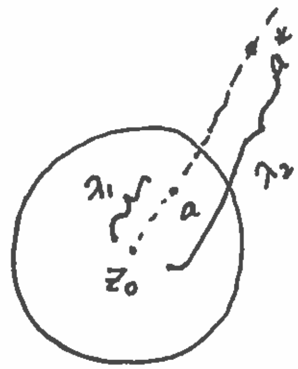
$$\boxed{a^* = \rho_K(a) = z_0 + \frac{r^2}{\bar{a} - \bar{z}_0}}$$

is the unique point $\overbrace{\;}^{a^*}$ on the line connecting z_0 and a s.t. $|z_0 - a| \cdot |z_0 - a^*| = r^2$

Note: $\rho_K(z_0) = \infty$, $\rho_K(\infty) = z_0$.

e.g. $z_0 = 0$; $r = 1$ gives

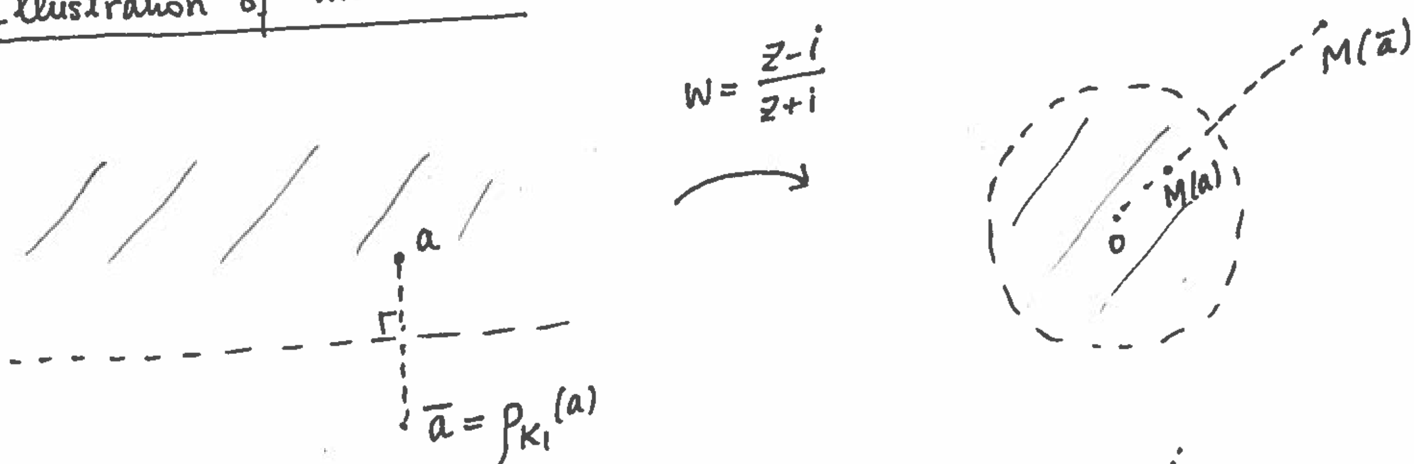
$$a^* = \frac{1}{\bar{a}}$$



$\lambda_1 \lambda_2 = r^2$

(reflections are anti-holomorphic – i.e., they preserve the angles but reverse the orientation.)

Illustration of the theorem. – $K_1 = \mathbb{R}$, $K_2 = C(0;1)$, $M =$ Cayley Transform



$$M(a) = \frac{a-i}{a+i} \quad ; \quad M(\rho_{K_1}(a)) = M(\bar{a}) = \frac{\bar{a}-i}{\bar{a}+i}$$

$$= 1 \Big/ \overline{\left(\frac{a-i}{a+i}\right)} = \rho_{K_2}(M(a))$$

Exercise. - Use this special case ($K_1 = \mathbb{R}$, $K_2 = C(0;1)$) to prove the theorem in general.

You will have to show that $z \mapsto \dfrac{az+b}{cz+d}$ preserves the upper half plane $\iff$ $a, b, c, d \in \mathbb{R}$.

§2. Some interesting corollaries. -

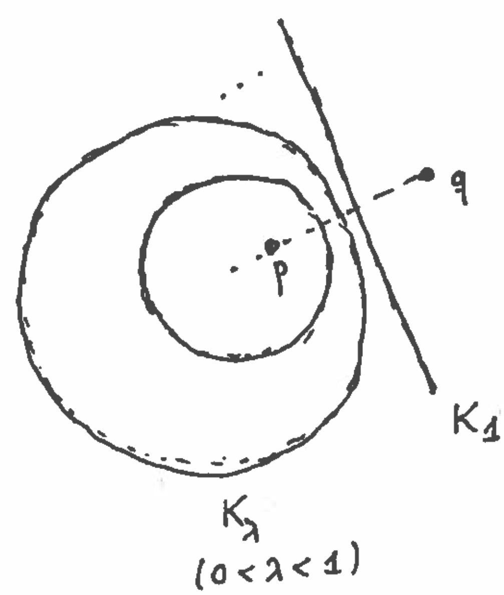
(1) Circles of Apollonius

Let $p, q \in \mathbb{C}$ be two distinct points., $\lambda \in (0,1)$, and define

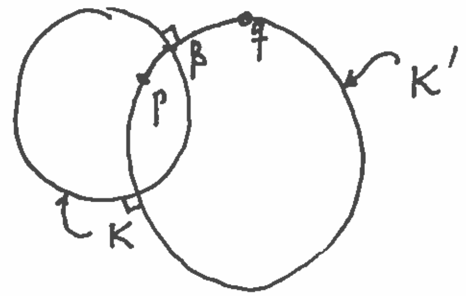
$$K_\lambda = \{ z : \; |z-p| = \lambda |z-q| \}.$$

Then $\rho_{K_\lambda}(p) = q \quad \forall \; \lambda \in (0,1]$ (note $\lambda = 1$, K_λ is the perpendicular bisector of the segment pq)

Proof. (hint: let
$$M(z) = \frac{z-p}{z-q},$$
so $M(p) = 0$; $M(q) = \infty$
$M(K_\lambda) = C(0;\lambda)$.)



(2) If $\rho_K(p) = q$, then any circle through p & q meets K at right angle.

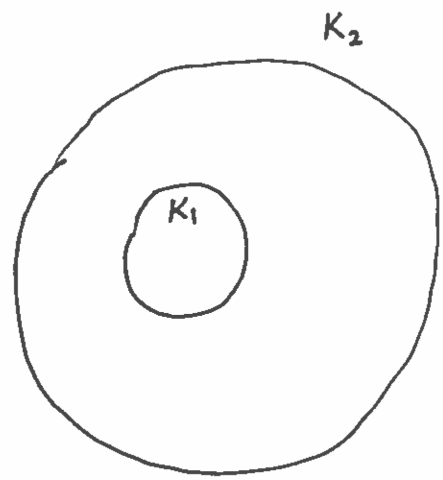


(Proof - hint: let β be a point of intersection of the two circles. Send β to ∞, using, e.g.

$$M(z) = \frac{1}{z-\beta}.)$$

(K : circle ; $p \in$ interior (K)
$q \in$ exterior (K))

(3) <u>Steiner's porism.</u>



Let K_1 and K_2 be two circles, K_1 in the interior of K_2.

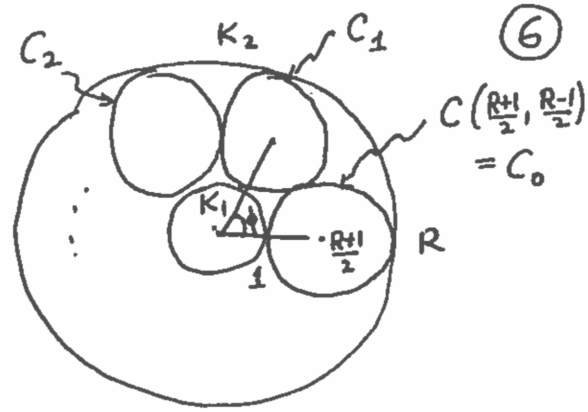
Then $\exists\ M \in \mathrm{Aut}(\hat{\mathbb{C}})$ s.t. $M(K_1)$ and $M(K_2)$ are concentric

<u>Proof</u>. - Show that $\exists!$ $\alpha \in$ Interior (K_1)
$\beta \in$ exterior (K_2) s.t. $\beta = \rho_{K_1}(\alpha) = \rho_{K_2}(\alpha)$.

(Hint: choose M s.t. $M(K_2) = i\mathbb{R}$
$M(K_1) = C(a;r)$; $r, a \in \mathbb{R}_{>0}$
$0 < r < a$.

then $\alpha = M^{-1}\left(\sqrt{a^2 - r^2}\right)$.)
(minus)

Now $z \mapsto \dfrac{1}{z-\beta}$ sends β to ∞ and hence α to the center of both $M(K_1)$ and $M(K_2)$. $\square$



Given a circle C_0 tangent to both K_1 and K_2, let C_1 be the unique (left - counterclockwise sense) circle tangent to three K_1, K_2, C_0 circles.

$K_1 = C(0;1)$

$K_2 = C(0;R)$

$$\phi = 4 \sin^{-1}\left(\sqrt{\frac{R-1}{4(R+1)}}\right)$$

(Exercise)

In the concentric circles K_1, K_2

$C_0 \longmapsto C_1$ is just a rotation by some angle ϕ.

- $\phi \in 2\pi\mathbb{Q} \iff$ rotation by ϕ has finite order (i.e. $r_\phi^N = Id$ for some $N \in \mathbb{N}$).

Hence, either $C_0 \mapsto C_1 \mapsto C_2 \mapsto \dots \mapsto C_N = C_0$ for any initial C_0

or $\{C_0, C_1, C_2, \dots\}$ is infinite (again for any initial C_0).

Steiner's porism is the same statement as above, for arbitrary K_1, K_2 - and follows from the concentric case since Möbius transformations can make K_1, K_2 concentric.