# Lecture 31

Our next main theorem is the following:

§1. Riemann Mapping Theorem (some texts call it the fundamental theorem of conformal geometry): Let $\Omega \subsetneq \mathbb{C}$ be a proper, open, connected and simply-connected set. Then $\Omega$ is conformally equivalent to the unit disc $\mathbb{D} = \{z : |z| < 1\}$.

Uniqueness of the conformal equivalence. There are a few conditions which guarentee uniqueness of conformal equivalence

$$f : \Omega \to \mathbb{D}$$

e.g.

- let $z_0 \in \Omega$. Then $\exists!\, f$ s.t. $f(z_0) = 0$ and $f'(z_0) \in \mathbb{R}_{>0}$.
- let $z_1, z_2, z_3 \in \Omega$, $w_1, w_2, w_3 \in \mathbb{D}$. Then $\exists!\, f$ s.t. $f(z_j) = w_j$ $(j = 1, 2, 3)$.

These uniqueness properties follow from the explicit description of $\mathrm{Aut}(\mathbb{D})$: if $f_1, f_2 : \Omega \xrightarrow{\sim} \mathbb{D}$ are two conformal equivalences then $f_2 \circ f_1^{-1} : \mathbb{D} \xrightarrow{\sim} \mathbb{D}$ must be of the form

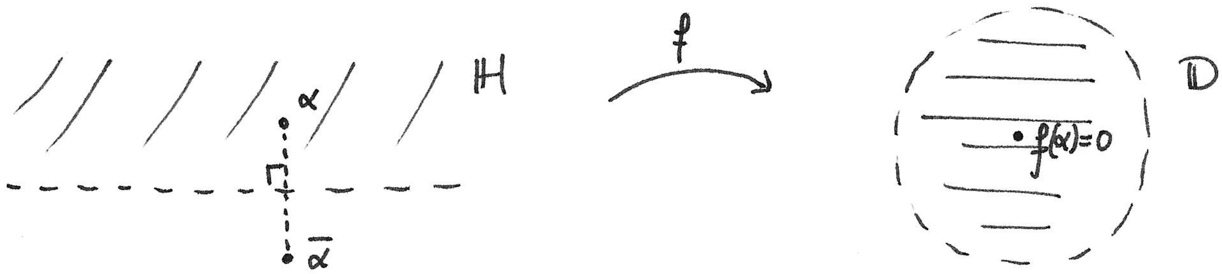
$$z \mapsto e^{i\phi} \frac{z-a}{1-\bar{a}z}.$$

e.g. - if $f_1$ and $f_2$ both map $z_0 \mapsto 0$ then $a = 0$ $(f_2 \circ f_1^{-1} : 0 \mapsto 0)$ and hence $f_1(z) = e^{i\phi} f_2(z)$. $f_1'(z_0), f_2'(z_0) \in \mathbb{R}_{>0} \Rightarrow \phi = 0$.

## §2. Glossary of some useful conformal equivalences –

**Prop.** Let $\mathbb{H} = \{\operatorname{Im} z > 0\}$ be the upper half plane.
Let $f : \mathbb{H} \to \mathbb{D}$ be a conformal equivalence. Then,
$\exists\ \alpha \in \mathbb{H}$ and $\lambda \in \partial\mathbb{D} = \{|z| = 1\}$ s.t. $f(z) = \lambda \cdot \dfrac{z-\alpha}{z-\bar{\alpha}}$.

Cayley transform: $C(z) = \dfrac{z-i}{z+i}$ $(\alpha = i,\ \lambda = 1)$.




Proof: Let $\alpha \in \mathbb{H}$ be s.t. $f(\alpha) = 0$. It is easy to see (from our description of $\operatorname{Aut}(\mathbb{D})$ or $\operatorname{Aut}(\mathbb{H})$) that $f$ is a Möbius transformation, hence commutes with reflections. Hence $f(\bar{\alpha}) = \infty$.

$$\Rightarrow \quad f(z) = \lambda \frac{z-\alpha}{z-\bar{\alpha}} \quad \text{for some } \lambda \in \mathbb{C} \setminus \{0\}.$$

As $|f(x)| = 1\ \ \forall x \in \mathbb{R}$, we get $|\lambda| = 1$ $\square$

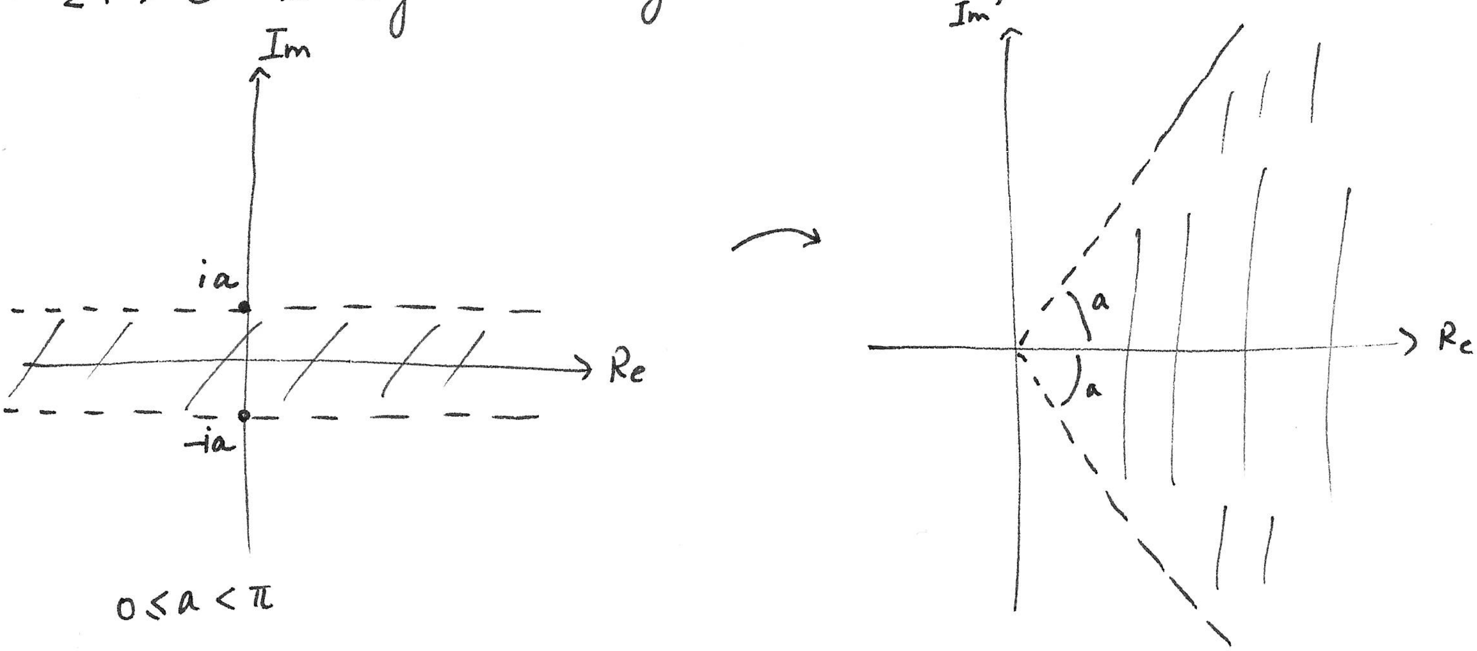
**Exercise.** – Let $M \in \operatorname{Aut}(\hat{\mathbb{C}})$ be a Möbius tranform. Show that $M = M_A$ for $A \in GL_2(\mathbb{C})$ with $\det(A) = 1$. Assume this $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Prove that $M$ stretches $\{z : |cz+d| < 1\}$ and

and shrinks $\{z : |cz+d| > 1\}$.

§3. Glossary cntd. – exponential function.

$$e^{x+iy} = e^{x} \cdot e^{iy} = e^{x}(\cos(y) + i\sin(y))$$

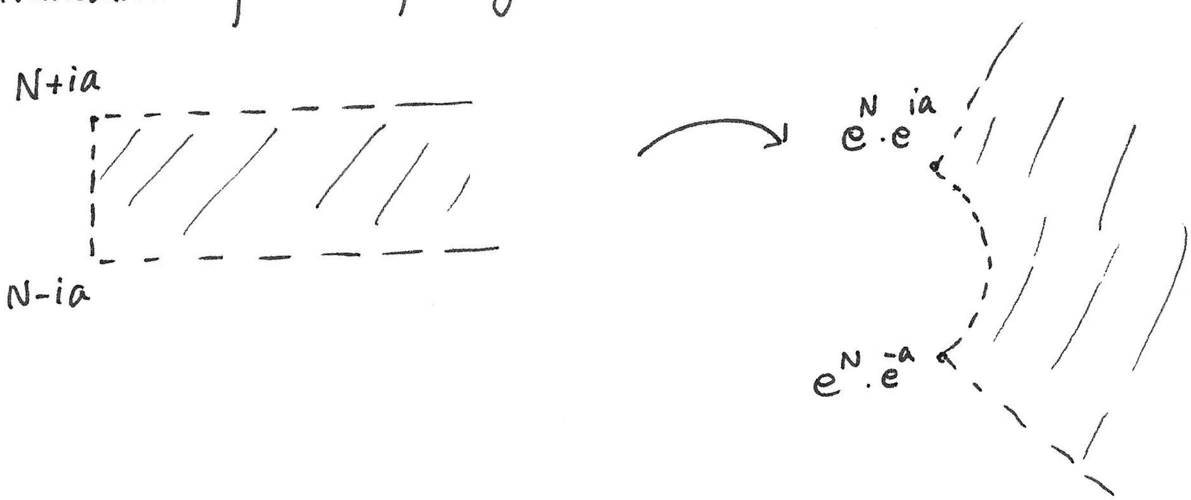
$z \mapsto e^{z}$ is bijective on any horizontal strip of width $< 2\pi$




$0 \leq a < \pi$

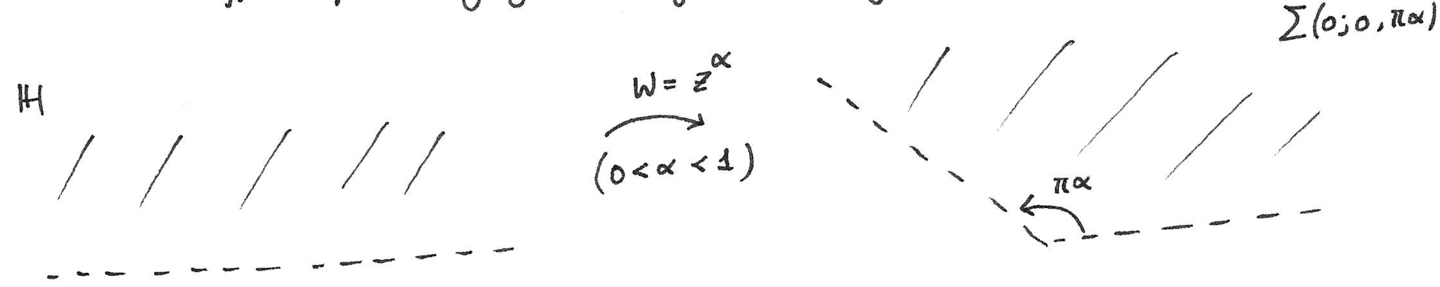
Sector $\Sigma(0; -a, a)$.
$= \{z : -a < \arg(z) < a\}$

Truncations of Real part give us



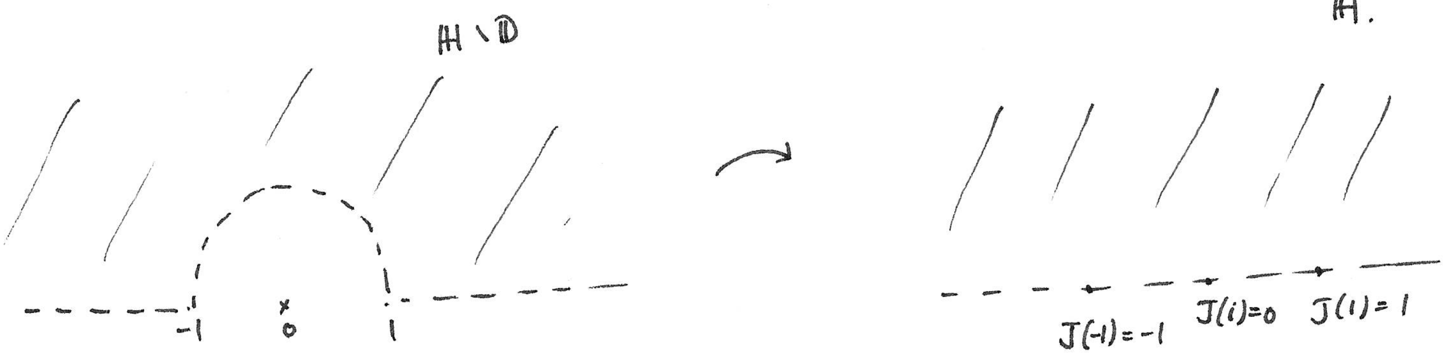

and so on!

Power function : $z \mapsto z^{\alpha} = e^{\alpha \log(z)}$ defined on a cut domain, has the effect of changing the angle of opening in a sector. eg :

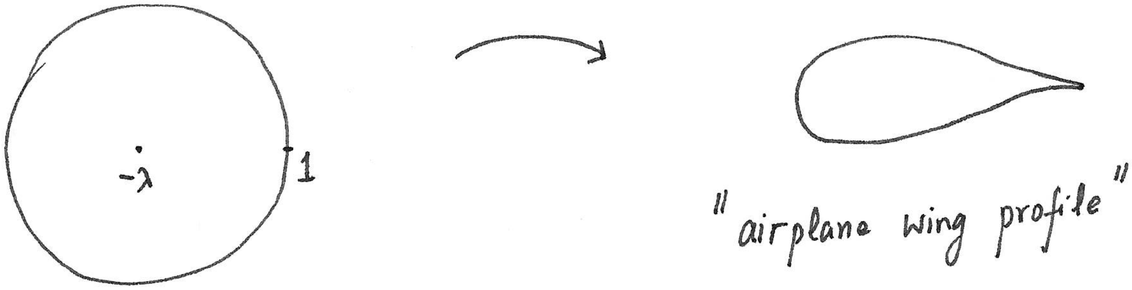



§4. ~~Joukowski~~ Joukowski or Zhoukovskii - type transformations.-

$J(z) = \dfrac{z+z^{-1}}{2}$ has the effect of mapping $\mathbb{H} \setminus \mathbb{D}$ to $\mathbb{H}$ conformally.




Remark. - $J(z) = \dfrac{z+z^{-1}}{2} = \cos(\phi)$ if $z = e^{i\phi}$ transforms $C(0;1) = \{z : |z|=1\}$ to segment $-1$ —— $1$. In general, for any circle $C_{\lambda} = \{z : |z+\lambda| = 1+\lambda\}$ $(0<\lambda<\infty)$, we get



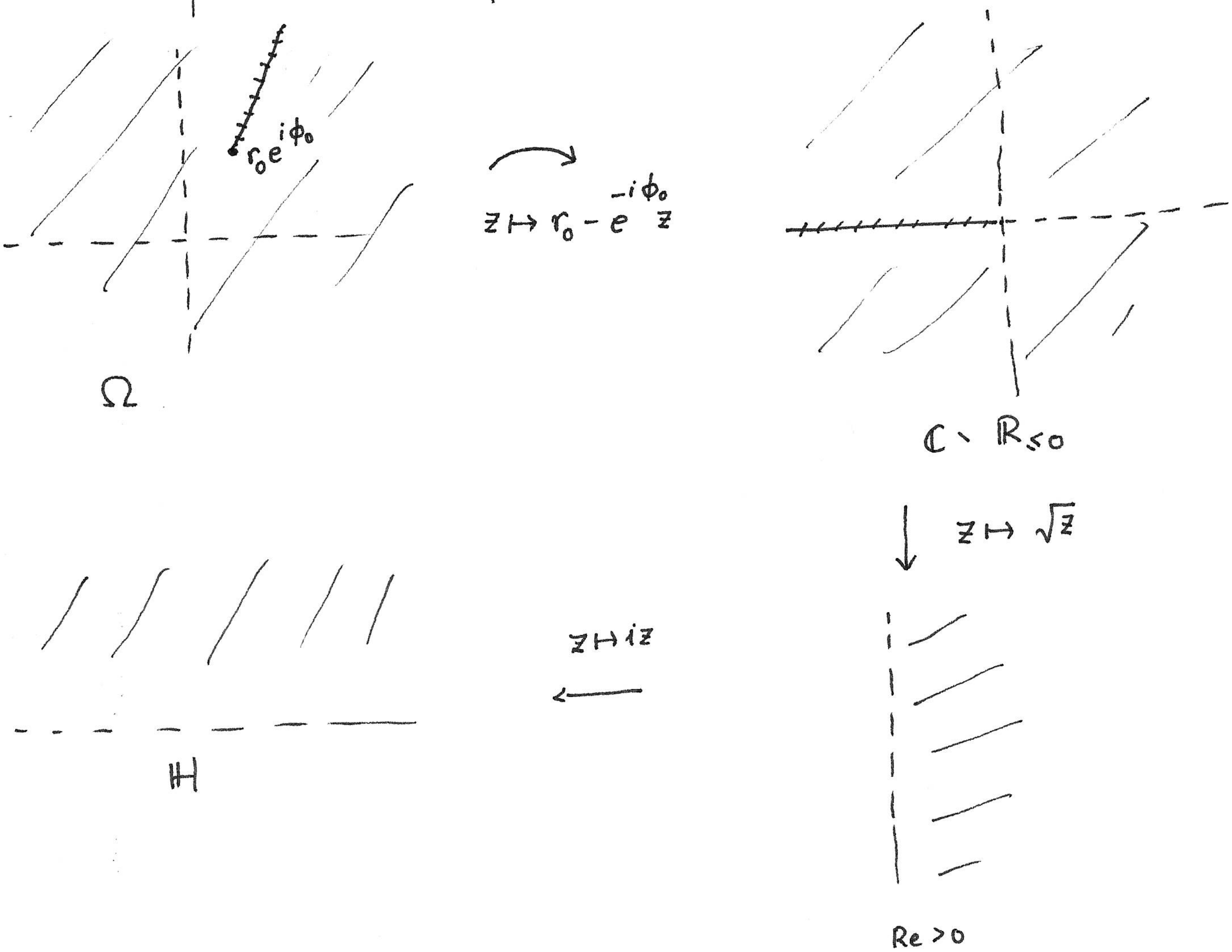

"airplane wing profile"

Exercise - Determine the conformal properties of

$$J_\lambda(z) = \lambda z + (1-\lambda) z^{-1}, \quad 0 < \lambda < 1.$$

## §5. Examples.

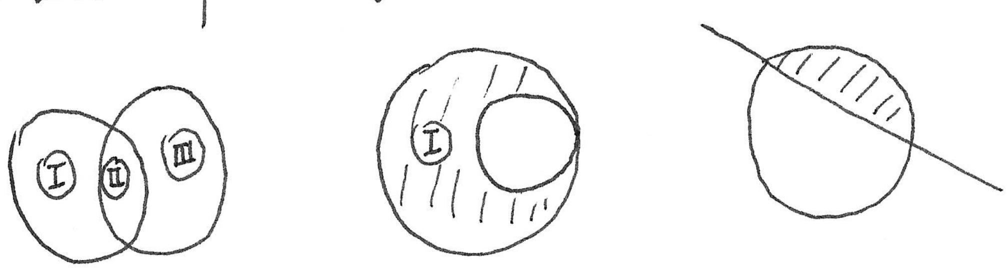
(1) Cut plane $\Omega = \mathbb{C} \setminus \mathbb{R}_{\geq 1} \cdot z_0$ $\qquad z_0 = r_0 e^{i\phi_0} \in \mathbb{C} \setminus \{0\}$.



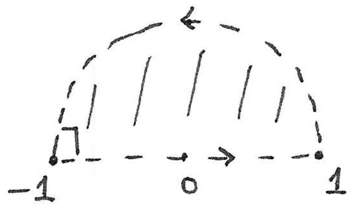

So, $\quad z \mapsto i\sqrt{r_0 - e^{-i\phi_0} z} \quad : \Omega \xrightarrow{\simeq} \mathbb{H}$.

(2) Circular lune shapes - regions enclosed by two arcs of circles (or lines)

e.g.

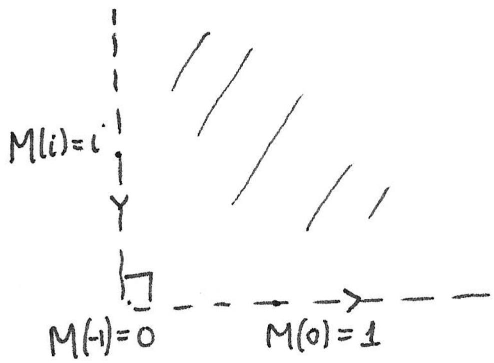


e.g. let $\Omega = \{ z : |z| < 1 \text{ and } \mathrm{Im}(z) > 0 \}$

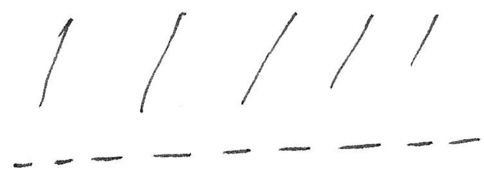



$\overset{M}{z \mapsto} \dfrac{1+z}{1-z}$

(Möbius transformation sending 1 to $\infty$)

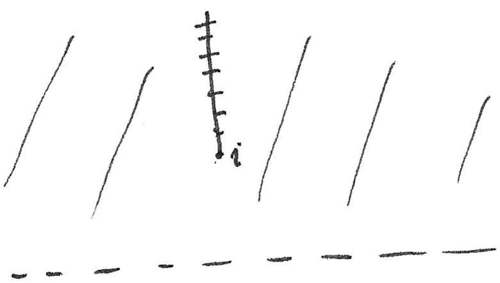



$z \mapsto z^2$

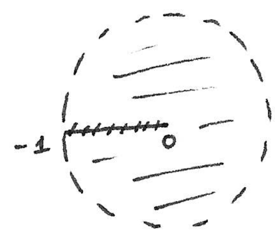


So, $z \mapsto \left( \dfrac{1+z}{1-z} \right)^2$ is a conformal equivalence between $\Omega$ & $\mathbb{H}$.

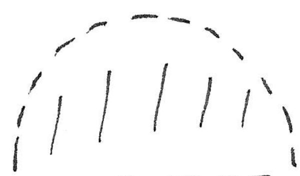
(3) $\Omega = \mathbb{H} \setminus \mathbb{R}_{\geq 1} i$




$(-1)\dfrac{z-i}{z+i}$



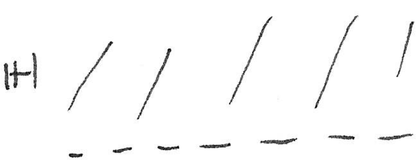

$i \cdot \sqrt{z}$



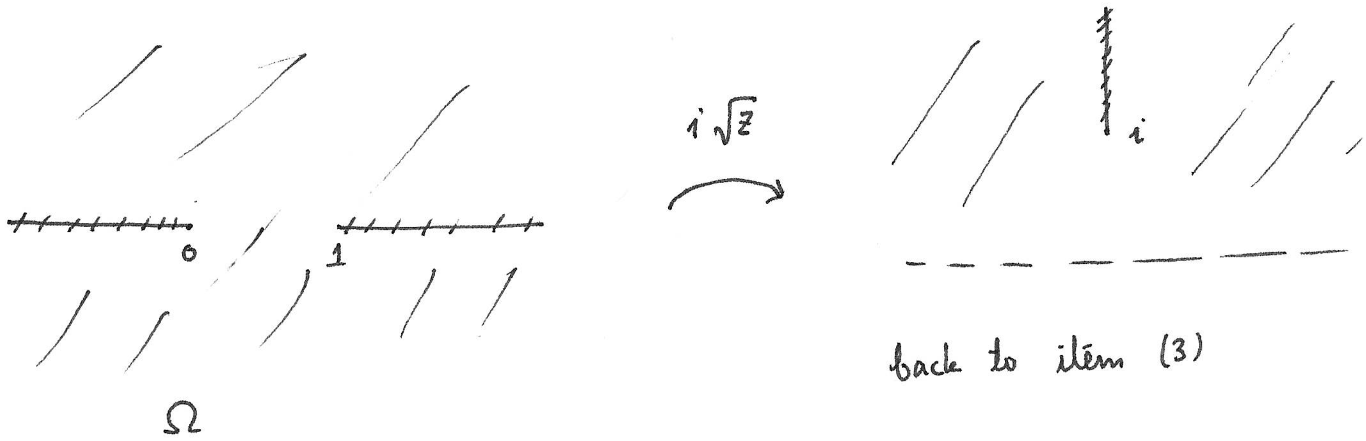
$z \mapsto \left( \dfrac{1+z}{1-z} \right)^2$

item (2) above

$z \mapsto \left( \dfrac{1 + i\sqrt{\frac{i-z}{i+z}}}{1 - i\sqrt{\frac{i-z}{i+z}}} \right)^2$



(4) Twice cut plane : $\Omega = \mathbb{C} \setminus (\mathbb{R}_{\geq 1} \cup \mathbb{R}_{\leq 0})$




back to item (3)

Exercise - generalize to $n$- cut plane.

(5) Half- strip. $\Omega = \{ z \mid \mathrm{Im}(z) > 0 ,\ \mathrm{Re}(z) \in (-\frac{\pi}{2}, \frac{\pi}{2}) \}$