Lecture 32

In this lecture, we give a proof of Riemann Mapping theorem, assuming the following results: (to be proved later).

(1) Montel's Theorem.- Let $\Omega \subseteq \mathbb{C}$ be an open, connected set, and $\mathcal{F}$ be a set of holomorphic functions of Ω satisfying the following (locally-bounded-ness) assumption:

$$\boxed{\forall \underset{\text{compact}}{K \subset \Omega}, \ \exists\, M_K \in \mathbb{R}_{>0} \text{ s.t. } |f(z)| < M_K \ \forall f \in \mathcal{F}, \forall z \in K}$$

Then, for any sequence of functions $(f_n)_{n=1}^{\infty}$ from $\mathcal{F}$, there exists a subsequence $n_1 < n_2 < \cdots$ s.t. $(f_{n_k})_{k=1}^{\infty}$ is uniformly convergent.

(2) Hurwitz' Theorem.- ($\Omega \subset \mathbb{C}$; open, connected). Assume $(f_n)_{n=1}^{\infty}$ is a uniformly convergent sequence of one-to-one holomorphic functions on Ω and $f = \lim_{n \to \infty} f_n$ is not constant. Then f is again one-to-one.

(3) Koebe's "stretching function".- Let $\Omega_0 \subsetneq \mathbb{D}$ proper, open, simply-connected set so that $0 \in \Omega_0$. Then, there is a one-to-one, holomorphic function $\kappa : \Omega_0 \to \mathbb{D}$ such that

$$\kappa(0) = 0 \quad \text{and} \quad |\kappa(z)| > |z|, \ \forall z \in \Omega_0 \setminus \{0\}.$$

§1. Proof of Riemann Mapping Theorem.

Let $\Omega \subsetneq \mathbb{C}$, a proper, open, connected & simply-connected set be given.

Step 1. Ω is conformally equivalent to $\Omega_0 \subset \mathbb{D}$ $(0 \in \Omega_0)$.

Proof. Let $\alpha \in \mathbb{C} \setminus \Omega$. $z - \alpha : \Omega \to \mathbb{C}$ is a nowhere vanishing hol. fn. on a simply connected domain, hence we can choose a single-valued branch of square-root

$$g(z) = \sqrt{z - \alpha} \quad : \Omega \to \mathbb{C}.$$

Note: (1) g is one-to-one $\left(g(z_1) = g(z_2) \Rightarrow g(z_1)^2 = g(z_2)^2 \right.$
$\left. \Rightarrow z_1 - \alpha = z_2 - \alpha \quad \Rightarrow z_1 = z_2. \right)$

(2) $w_0 \in g(\Omega) \Rightarrow -w_0 \notin g(\Omega)$. (same argument as above)
$(w_0 \neq 0)$

Pick $w_0 \in g(\Omega)$. As $g(\Omega)$ is open (open-mapping theorem), we can find $r > 0$ s.t. $D(w_0; r) \subset \Omega$

$\overline{D(-w_0; r)} \cap \Omega = \phi$. (see (2))

i.e. $|g(z) + w_0| > r$, $\forall z \in \Omega$. Let $h(z) = \dfrac{r/2}{g(z) + w_0}$

Hence, $h(z) - h(z_0) : \Omega \longrightarrow \mathbb{D}$ is one-to-one and
$z_0 \longmapsto 0.$

So $\Omega \cong$ Image of $z \mapsto h(z) - h(z_0) = \Omega_0 \subset \mathbb{D}$ as claimed $\square$

Step 2. - An optimization problem. - We will now assume $\Omega_0 \subset \mathbb{D}$ (simply-connected domain) is given and $0 \in \Omega_0$.

Let $\mathcal{F} = \{ f : \Omega_0 \to \mathbb{D}$, one-to-one hol. fns s.t. $f(0) = 0 \}$. $\left(\Omega_0 \overset{j}{\hookrightarrow} \mathbb{D} \in \mathcal{F}, \text{ so } \mathcal{F} \neq \phi \right)$

Choose $a_0 \in \Omega_0$; $a_0 \neq 0$, and let $A = \sup \{ |f(a_0)| : f \in \mathcal{F} \} \in (0, 1]$

$(|j(a_0)| = |a_0| > 0$, so, $A > 0.)$

Claim: $\exists \; h \in \mathcal{F}$ s.t $|h(a_0)| = A$.

Proof: Let $(f_n)_{n=1}^{\infty}$ be a seq. of hol. fns. from $\mathcal{F}$ s.t. $\lim_{n \to \infty} |f_n(a_0)| = A$.

By Montel's Thm. $\exists \; (f_{n_k})_{k=1}^{\infty} \to h$ uniformly. Thus,

h is again holomorphic (by Weierstrass' Thm.), non-constant $h(a_0) \neq h(0) = 0$.

hence one-to-one (by Hurwitz' Thm).

So, $h \in \mathcal{F}$ and $|h(a_0)| = \lim_{k \to \infty} |f_{n_k}(a_0)| = A$.

Last Step. - $h : \Omega_0 \to \mathbb{D}$ as above. We will show that $h(\Omega_0) = \mathbb{D}$.

If $h(\Omega_0) \subsetneq \mathbb{D}$, then we have $\kappa : h(\Omega_0) \to \mathbb{D}$ s.t. $\kappa(0) = 0$ and $|\kappa(z)| > |z| \;\; \forall z \neq 0$.

$\Rightarrow$ $\kappa \circ h \in \mathcal{F}$ from Step 2 above,

and $|\kappa \circ h(a_0)| = |\kappa(h(a_0))| > |h(a_0)| = A$

a contradiction. $\square$

§2. Proof of Hurwitz' Theorem. –

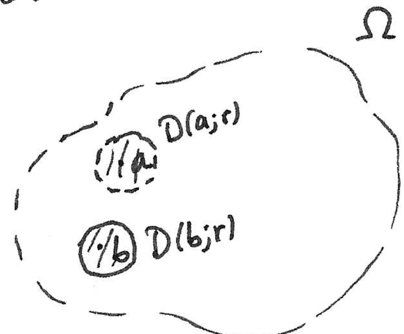
Given $\left(f_n : \Omega \to \mathbb{C} \text{ hol. one-to-one} \right)_{n=1}^{\infty} \longrightarrow f$ converging uniformly,

and $f : \Omega \to \mathbb{C}$ is not constant.

To prove: f is one-to-one.

Assume f is not one-to-one. Let $a \neq b$, $a, b \in \Omega$ be such that $f(a) = f(b) = w_0$. Let $r > 0$ be small enough so that

$D(a,r) \cap D(b,r) = \phi$, and $\overline{D(a,r)}$, $\overline{D(b,r)} \subset \Omega$.



(*) # Solns. to $f(z) = w_0$ within $D(a;r)$ or $D(b;r)$ $< \infty$ since f is not constant

≥ 1 since $f(a) = f(b) = w_0$.

$$\Rightarrow \quad 1 \leq \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z) - w_0}\, dz = \lim_{n \to \infty} \frac{1}{2\pi i} \int_C \frac{f_n'(z)}{f_n(z) - w}\, dz$$

$(C = C(a;r)$ or $C(b;r))$

$\Rightarrow \exists\, N > 0$ s.t. $f_N(z) = w_0$ has at least one solution within $C(a;r)$ and one within $C(b;r)$ – contradicting univalence of f_N.

$\square$

Note: (*) fails if $\lim_{n \to \infty} f_n$ is constant e.g. $f_n = \frac{z}{n}$, $f \equiv 0$.

§3. Proof of Koebe's Theorem (Paul Koebe 1882-1945).

$\Omega_0 \subsetneq \mathbb{D}$ simply-connected domain ; $0 \in \Omega_0$. Goal is to find a univalent, holomorphic $K : \Omega_0 \to \mathbb{D}$ s.t. $K(0) = 0$ and $|K(z)| > |z|$ $\forall z \in \Omega_0 \setminus \{0\}$.

Choose $a \in \mathbb{D} \setminus \Omega_0$. Define $\varphi_a = \dfrac{z-a}{1-\bar{a}z} \in \mathrm{Aut}(\mathbb{D})$

$\varphi_a(z) \neq 0$ $\forall z \in \Omega_0$
Ω_0 is simply-connected $\Rightarrow$ we have a hol. square root

$$g(z) = \sqrt{\frac{z-a}{1-\bar{a}z}} : \Omega_0 \to \mathbb{D}$$

univalent.

Now, let $b = g(0)$ and define $K(z) = \varphi_b(g(z))$.

- K is univalent since φ_b and g are
- $K(0) = \varphi_b(b) = 0$.
- $|K(z)| > |z|$ for $z \neq 0$. Let $\Omega_1 = K(\Omega_0)$, so that

$K : \Omega_0 \xrightarrow{\sim} \Omega_1 \subset \mathbb{D}$. Let $h : \Omega_1 \to \Omega_0$ be its inverse

$$h = K^{-1} = g^{-1} \circ \varphi_b^{-1} = g^{-1} \circ \varphi_{-b} \quad ; \text{ and}$$

$$g^{-1}(z) = \varphi_{-a}(z^2) = \frac{z^2+a}{1+\bar{a}z^2}.$$ So, $h = K^{-1}$ is defined on $\mathbb{D}$, and is not a rotation. By Schwarz' lemma, $|h(z)| < |z|$ $\forall z \neq 0$

Set $z = K(w)$ to get $|w| < |K(w)|$ $\forall w \in \Omega_0 \setminus \{0\}$. □