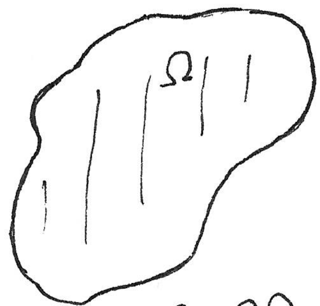
Lecture 33

§1. Dirichlet's[*] boundary value problem. -

Let C be a contour in $\mathbb{C}$ and

Ω = interior (C).



$C = \partial\Omega$ is a contour

Let a continuous, $\mathbb{R}$-valued function

$g : C \to \mathbb{R}$ be given.

Problem. - Find $u : \Omega \to \mathbb{R}$ harmonic s.t. $\forall z_0 \in C$

$$\lim_{\substack{z \to z_0 \\ (z \in \Omega)}} u(z) = g(z_0).$$

In other words, u extends to a continuous function $u : \overline{\Omega} \to \mathbb{R}$, and $u\big|_{\partial\Omega} = g$.

§2. $[\Omega, C, g$: as above$]$

Optimization problem. - Let $\mathcal{F}_g = \left\{ f \in C^2(\Omega; \mathbb{R}) : \begin{array}{l} f \text{ extends to} \\ \text{a cnts. } \overline{\Omega} \to \mathbb{R} \\ \text{s.t. } f\big|_{\partial\Omega} = g \end{array} \right\}$

Define $D : \mathcal{F}_g \longrightarrow \mathbb{R}_{\geq 0}$ by

$$D[f] = \iint_{\Omega} (f_x^2 + f_y^2)\, dx\, dy \qquad \left(\begin{array}{l}\text{Dirichlet's} \\ \text{energy functional}\end{array}\right)$$

Find $u \in \mathcal{F}_g$ s.t. $D[u] = \inf \{ D[v] : v \in \mathcal{F}_g \}$.

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[*] Lejeune Dirichlet (1805-1859)

§3. <u>Lemma</u>.- Let $u : \Omega \to \mathbb{R}$ be a solution to the optimization problem (§2). Then u is harmonic - hence ~~a~~ the solution to the boundary value problem.

(Note - if $u_1, u_2 : \Omega \to \mathbb{R}$ are two harmonic functions whose boundary values $= g$, then $u_1 - u_2$ is a harmonic fn vanishing at $\partial\Omega$. By max. modulus principle, $u_1 - u_2 \equiv 0$ on Ω. This shows uniqueness of solution to the boundary value problem.)

Proof. [Variational method.] Let $\varphi \in C^2(\Omega; \mathbb{R})$ be such that $\underset{\text{cnts}}{\varphi : \overline{\Omega} \to \mathbb{R}}$ and $\varphi\big|_{\partial\Omega} \equiv 0$. Then, $\forall\, t \in \mathbb{R}$, $u + t\varphi \in \mathcal{F}_g$.

$$D[u+t\varphi] = \iint_\Omega \left((u_x + t\varphi_x)^2 + (u_y + t\varphi_y)^2 \right) dx\, dy$$

$$= D[u] + 2t \iint_\Omega (u_x \varphi_x + u_y \varphi_y)\, dA + t^2 D[\varphi]$$

$t \mapsto D[u+t\varphi]$ has min. at $t=0$. So, its derivative at $t=0$ is 0

i.e.,
$$\iint_\Omega (u_x \varphi_x + u_y \varphi_y)\, dA = 0$$

By Green's theorem:

$$\int_{\partial\Omega} (u_x \varphi)\, dy - (u_y \varphi)\, dx = \iint_{\Omega} \left(\frac{\partial}{\partial x}(u_x \varphi) + \frac{\partial}{\partial y}(u_y \varphi) \right) dA$$

$$= \iint_{\Omega} (u_{xx} + u_{yy})\, \varphi \, dA + \iint_{\Omega} (u_x \varphi_x + u_y \varphi_y)\, dA.$$

As $\varphi \equiv 0$ on $\partial\Omega$, and $\iint_{\Omega} u_x \varphi_x + u_y \varphi_y \, dA = 0$ as we proved above, we get

$$\iint_{\Omega} (u_{xx} + u_{yy})\, \varphi \, dA = 0 \quad \forall\, \varphi \in C^2(\Omega; \mathbb{R}),\ \varphi \equiv 0 \text{ on } \partial\Omega.$$

$$\Rightarrow \quad u_{xx} + u_{yy} = 0, \quad \text{i.e. } u \text{ is harmonic} \qquad \square$$

§4. Remark.- Dirichlet's boundary value problem can be solved for any boundary data $g : \partial\Omega \to \mathbb{R}$. However, the resulting solution may have infinite Dirichlet energy. Such examples are due to Weierstrass and Hadamard.

Example. Let $\Omega = \mathbb{D}$, $C = \partial\Omega = \{e^{it} : t \in [0, 2\pi]\}$

$$g(e^{it}) = \sum_{n=1}^{\infty} \frac{\cos(n!\,t)}{n^4}$$

$$= \text{real part of} \quad z \mapsto \sum_{n=1}^{\infty} \frac{z^{n!}}{n^4} =: f(z)$$

So, $u : \Omega \to \mathbb{R}$ is $\displaystyle u(r.e^{it}) = \sum_{n=1}^{\infty} r^{n!} \frac{\cos(n!\,t)}{n^4}$

Note $\displaystyle D[u] = \iint_{\Omega} |f'(z)|^2 \, dA$ if $u = \operatorname{Re} f$

$$= \pi \sum_{n=1}^{\infty} n\, |c_n|^2 \qquad \text{if } f(z) = \sum_{n=0}^{\infty} c_n z^n \text{ on } \mathbb{D}$$

Hence, in our case $\displaystyle D[u] = \pi \sum_{n=1}^{\infty} \frac{n!}{n^8} = \infty.$

$\square$

§5. Dirichlet's boundary value problem on ~~a~~ the unit disc – Poisson's kernel.

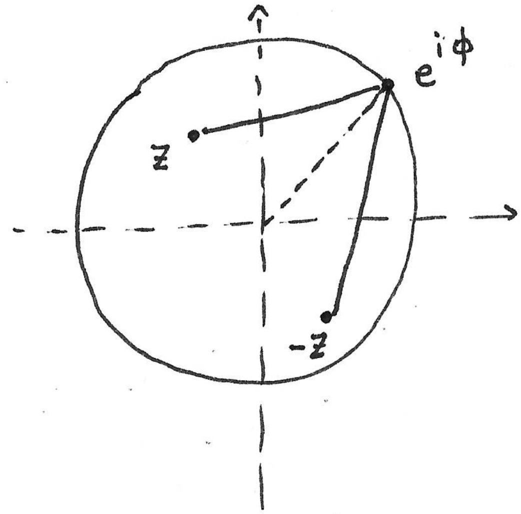
Define
$$\boxed{P(z; \phi) = \frac{e^{i\phi} + z}{e^{i\phi} - z}} \qquad z \in \mathbb{D},\ \phi \in \mathbb{R} \pmod{2\pi}$$

Simeon Denis Poisson (1781–1840)

$$K(z;\phi) := \operatorname{Re} P(z;\phi)$$

$$= \operatorname{Re} \frac{(e^{i\phi}+z)(e^{-i\phi}-\bar{z})}{|e^{i\phi}-z|^2}$$

$$= \operatorname{Re} \frac{1-|z|^2+ze^{-i\phi}-e^{i\phi}\bar{z}}{1+|z|^2-2|z|\cos(\theta-\phi)}$$



$$P(z;\phi) = \frac{e^{i\phi}+z}{e^{i\phi}-z}$$

Poisson kernel

So $$\boxed{K(r\cdot e^{i\theta};\phi) = \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2}}$$

<u>Theorem</u>. - Let $g : \partial\mathbb{D} \to \mathbb{R}$ be a continuous function.
(written as $g(\phi)$, $\phi \in \mathbb{R}$ s.t. $g(\phi+2\pi) = g(\phi)$.)

Then $u(z) := \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} K(z;\phi)\, g(\phi)\, d\phi$ is a harmonic function on $\mathbb{D}$, and $\displaystyle\lim_{r\to 1^-} u(re^{i\phi}) = g(\phi) \;\; \forall \phi$.

§6. Derivation of Poisson's kernel - I. - using power series.

Let $u : \mathbb{D} \to \mathbb{R}$ be a harmonic function which extends to $\overline{\mathbb{D}} \xrightarrow[\text{cnts}]{} \mathbb{R}$.

Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function s.t. $u = \operatorname{Re} f$.

Then $\quad f(z) = \sum_{n=0}^{\infty} a_n z^n \quad$ (Taylor series expansion of f)
[changing f to $f - i\operatorname{Im}(a_0)$, we may assume $a_0 \in \mathbb{R}$]

$$u(z) = \operatorname{Re} f(z) = \frac{1}{2}\left(f(z) + \overline{f(z)}\right)$$

$$= a_0 + \sum_{n=1}^{\infty} \frac{a_n z^n + \bar{a}_n (\bar{z})^n}{2}$$

$$= a_0 + \frac{1}{2} \sum_{n=1}^{\infty} r^n \left(a_n e^{in\phi} + \bar{a}_n e^{-in\phi}\right) \quad \text{if } z = re^{i\phi}.$$

Using $\displaystyle\int_0^{2\pi} e^{ik\theta}\, d\theta = \begin{cases} 0 & \text{if } k \neq 0 \\ 2\pi & \text{if } k = 0 \end{cases}$, we get

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} u(\rho e^{i\phi})\, d\phi \quad ; \quad a_n = \frac{1}{\pi} \int_0^{2\pi} \frac{u(\rho e^{i\phi})}{\rho^n} \cdot e^{-in\phi}\, d\phi .$$

here, we can take $0 < \rho < 1$ as large as we want. Letting $\rho \to 1$ and writing $g(\phi) = u(e^{i\phi})$, we obtain:

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} g(\phi)\, d\phi \quad ; \quad a_n = \frac{1}{\pi} \int_0^{2\pi} g(\phi) e^{-in\phi}\, d\phi .$$

$$\Rightarrow \quad f(z) = \sum_{n=0}^{\infty} a_n z^n = \frac{1}{2\pi}\int_0^{2\pi} g(\phi)\, d\phi + \sum_{n=1}^{\infty} \frac{1}{\pi}\int_0^{2\pi} g(\phi) e^{-in\phi} z^n\, d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} g(\phi)\cdot\left\{1 + 2\cdot\sum_{n=1}^{\infty} e^{-in\phi} z^n\right\} d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} g(\phi)\cdot \underbrace{\left(1 + \frac{2\cdot e^{-i\phi} z}{1 - e^{-i\phi} z}\right)}_{P(z;\phi)\ \leftarrow\ \text{Poisson kernel.}} d\phi$$

§7. Derivation of Poisson kernel - II. - using Möbius transformations

Recall: mean value property of harmonic functions implies

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\phi})\, d\phi$$

Let $z_0 \in \mathbb{D}$ and consider the Möbius transformation

$$T(z) = \frac{z + z_0}{1 + \bar{z}_0 z} \qquad (T(0) = z_0).$$

Let $w(z) = u(T(z))$ - again a harmonic fn. - hence
[exercise - check this]

$$w(0) = \frac{1}{2\pi}\int_0^{2\pi} w(e^{i\phi})\,d\phi \quad , \text{ i.e.,}$$

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u\left(\frac{e^{i\phi}+z_0}{1+\bar{z}_0 e^{i\phi}}\right) d\phi \quad - \quad (*)$$

Change of variables $\quad e^{i\psi} = \dfrac{e^{i\phi}+z_0}{1+\bar{z}_0 e^{i\phi}}$

so, $\quad e^{i\phi} = \dfrac{e^{i\psi}-z_0}{1-\bar{z}_0 e^{i\psi}}$

Exercise: $\quad d\phi = \dfrac{1-|z_0|^2}{|e^{i\psi}-z_0|^2}\,d\psi$. Hence, from (*) above,

we get

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\psi}) \cdot \underbrace{\frac{1-|z_0|^2}{|e^{i\psi}-z_0|^2}}_{\substack{\text{Poisson kernel } K(z_0;\psi) \\ = \operatorname{Re} P(z_0;\psi)}} \cdot d\psi$$

□