Lecture 34

Recall. last time we defined the Poisson kernel

$$P(z;\phi) = \frac{e^{i\phi}+z}{e^{i\phi}-z} \quad ; \quad z \in \mathbb{D}, \ \phi \in \mathbb{R} \bmod 2\pi\mathbb{Z}.$$

$$K(z;\phi) = \operatorname{Re} P(z,\phi) = \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2} \quad \text{if } z = re^{i\theta}$$

$$= \frac{1-|z|^2}{|e^{i\phi}-z|^2}$$

§1. Properties of $K(z;\phi)$

(1) Positivity : $K(z;\phi) > 0 \quad \forall z \in \mathbb{D}, \phi \in \mathbb{R}$

(2) Total mass = 1 : $\displaystyle \frac{1}{2\pi}\int_0^{2\pi} K(z;\phi)\,d\phi = 1 , \quad \forall z \in \mathbb{D}.$

Proof. – $$\frac{1}{2\pi}\int_0^{2\pi} K(z,\phi)\,d\phi = \operatorname{Re} \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\phi}+z}{e^{i\phi}-z}\,d\phi$$

$$\left(s = e^{i\phi} \Rightarrow ds = i\cdot e^{i\phi}\,d\phi \Rightarrow d\phi = \frac{1}{i} s^{-1} ds \right)$$

$$= \operatorname{Re} \frac{1}{2\pi i}\int_{C(0;1)} \frac{s+z}{(s-z)s}\,ds = 1.$$

C(0;1)

by Cauchy's formula

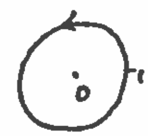
(3) If $z = e^{i\psi}$, then $K(z,\phi) = 0 \quad \forall \phi \neq \psi$.

§2. <u>Theorem</u>. Given a cnts. real-valued function $g(\phi)$ on $\partial\mathbb{D}$ (i.e. $g(\phi) = g(\phi + 2\pi)$), $u : \mathbb{D} \to \mathbb{R}$ defined by

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} K(z,\phi)\, g(\phi)\, d\phi$$

is harmonic, and $\forall\, \theta_0 \in [0, 2\pi]$, $\displaystyle\lim_{\substack{z \to e^{i\theta_0} \\ z \in \mathbb{D}}} u(z) = g(\theta_0)$.

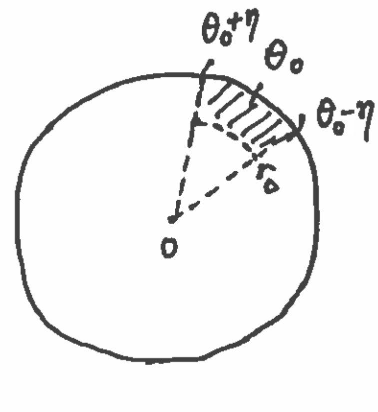
<u>Proof</u>.- As $u(z) = \operatorname{Re} f(z)$, $\displaystyle f(z) = \frac{1}{2\pi i} \int \frac{s+z}{s-z}\, g(\arg(s))\, \frac{ds}{s}$



and f is holomorphic (by our general result on functions defined by integrals), we get that u is harmonic.

Now, let $\theta_0 \in [0, 2\pi)$ and $\varepsilon > 0$ be given. We have to find $r_0 \in (0,1)$ s.t. $|u(z) - g(\theta_0)| < \varepsilon \quad \forall\, z = re^{i\theta}$ s.t. $r_0 \leq r < 1$, $\theta \in (\theta_0 - \delta, \theta_0 + \delta)$

$\cancel{\eta\ \delta > 0}$ $\delta > 0$



[in the end, $\delta = \frac{\eta}{2}$]

We choose η so that

$$|g(\theta) - g(\theta_0)| < \frac{\varepsilon}{2} \qquad - (1)$$

$$\forall\, \theta \in (\theta_0 - \eta, \theta_0 + \eta)$$

Since $g(\theta_0) = \frac{1}{2\pi}\int_0^{2\pi} K(z,\phi)\, g(\theta_0)\, d\phi$, by property (2) of §1 above,

we can write
$$u(z) - g(\theta_0) = \frac{1}{2\pi}\int_0^{2\pi} K(z,\phi)\,\big(g(\phi)-g(\theta_0)\big)\, d\phi .$$

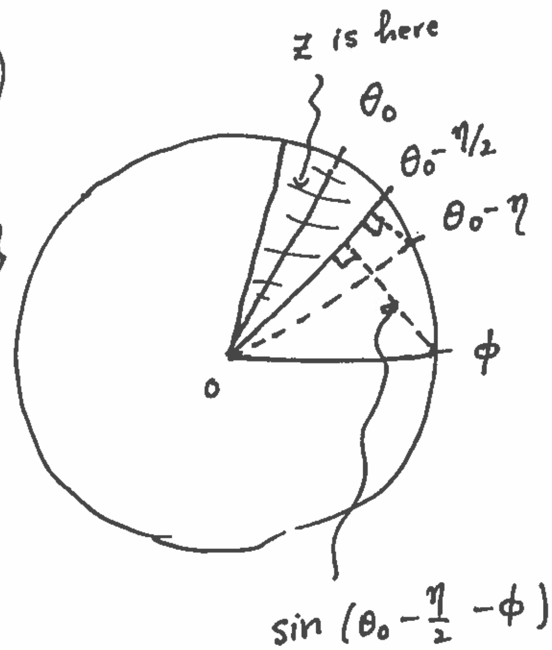
(1)
$$\left| \frac{1}{2\pi}\int_{|\phi-\theta_0|<\eta} K(z,\phi)\,\big(g(\phi)-g(\theta_0)\big)\, d\phi \right|$$

$$\leq \frac{1}{2\pi}\int_{|\phi-\theta_0|<\eta} K(z,\phi)\,\big|g(\phi)-g(\theta_0)\big|\, d\phi \qquad (K \text{ is positive})$$

$$\leq \frac{\varepsilon}{2}\cdot\left(\frac{1}{2\pi}\int_0^{2\pi} K(z,\phi)\, d\phi\right) = \frac{\varepsilon}{2} \qquad (\text{total mass} = 1).$$

(2) To estimate $K(z,\phi)$ when $\arg(z)$ is close to θ_0 (say $|\theta-\theta_0| < \frac{\eta}{2}$) and $|\phi-\theta_0| \geq \eta$, we see easily that

$$\left| re^{i\theta} - e^{i\phi} \right| \geq \sin\left(\eta/2\right)$$



(Proof $\operatorname{Min}\left\{ |z - e^{i\phi}| : \begin{array}{l} z \in \mathbb{D} \\ \arg(z) \in \left(\theta_0 - \frac{\eta}{2}, \theta_0 + \frac{\eta}{2}\right) \end{array} \right\}$

(assume $\phi < \theta_0$)
$$= \sin\left(\theta_0 - \frac{\eta}{2} - \phi\right)$$

whose smallest value, for $|\phi - \theta_0| > \eta$

is when $\phi = \theta_0 - \eta$. $\square$)

So, $$\left| \frac{1}{2\pi} \int_{|\phi-\theta_0|\geq\eta} K(z;\phi)\,(g(\phi)-g(\theta_0))\,d\phi \right| \leq \frac{M(1-|z|^2)}{\sin(\eta/2)} \qquad (2)$$

$\left(M = \operatorname{Max}\{|g(\phi)-g(\theta_0)| : \phi \in [0,2\pi]\} \right)$.

Pick $r_0 \in (0,1)$ s.t. $\quad 1-r^2 < \dfrac{\sin(\eta/2)}{M} \cdot \dfrac{\varepsilon}{2} \quad \forall\, r \in (r_0,1)$.

Combining (1) and (2) we get

$$|u(z) - g(\theta_0)| < \varepsilon \quad \forall\, z \text{ s.t. } \begin{array}{l} |z| \in (r_0, 1) \\ \arg(z) \in \left(\theta_0 - \frac{\eta}{2}, \theta_0 + \frac{\eta}{2}\right). \end{array}$$

$\square$

§3. Remarks.-

(1) The Poisson Kernel gives rise to a family of probability measures on S^1, parametrized by $z \in \mathbb{D}$.

$$\mu_z(A) := \frac{1}{2\pi} \int_A K(z,\phi)\,d\phi \qquad A \subset [0,2\pi] \text{ Borel set.}$$

At $z=0$, $K(0,\phi)=1 \Rightarrow \mu_0 =$ Lebesgue measure.

Fact: Lebesgue measure is the unique rotational invariant measure on S^1.

Rotations $z \mapsto \lambda z$, $|\lambda| = 1$ form a subgroup of $\mathrm{Aut}(\mathbb{D})$ consisting of conformal automorphisms fixing $0 \in \mathbb{D}$.

$\mathrm{Stab}(z_0) := \{ f \in \mathrm{Aut}(\mathbb{D})$ such that $f(z_0) = z_0 \}$. $(z_0 \in \mathbb{D})$.

It follows from the derivation (II) of Poisson kernel (from last lecture) that

μ_{z_0} is the unique Probability measure on S^1 which is invariant under $\mathrm{Stab}(z_0)$

(2) Relation with hyperbolic metric

Definition : For $\gamma : [0,1] \to \mathbb{D}$, define <u>the hyperbolic length</u> of γ

$$l_H(\gamma) := \int_0^1 \frac{|\gamma'(t)|}{1 - |\gamma(t)|^2} \, dt.$$

<u>Lemma</u>. - l_H is conformally invariant (i.e. invariant under $\mathrm{Aut}(\mathbb{D})$)

Proof. - For $T \in \mathrm{Aut}(\mathbb{D})$, $T(z) = e^{i\theta} \dfrac{z - \alpha}{1 - \bar{\alpha} z}$, $\alpha \in \mathbb{D}$, $\theta \in \mathbb{R}$,

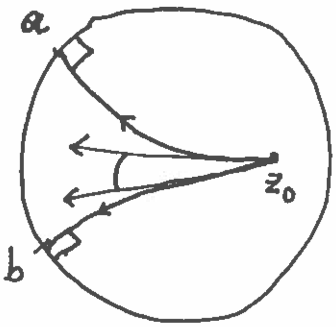
one can easily show that $\dfrac{|T'(z)|}{1 - |T(z)|^2} = \dfrac{1}{1 - |z|^2}$ $\square$

<u>Definition</u> $g:[0,1]\to\mathbb{D}$ $(g(0)=\alpha,\ g(1)=\beta)$ is a geodesic joining α and β if

$$\ell_H(g) = \text{MIN}\{\ell_H(\gamma) \mid \gamma:[0,1]\to\mathbb{D},\ \gamma(0)=\alpha,\ \gamma(1)=\beta\}$$

<u>Prop.</u> – (1) Given $\alpha,\beta\in\mathbb{D}$, there is a unique geodesic passing through α and β which is the unique circle through α,β meeting $\partial\mathbb{D}$ at right angles.

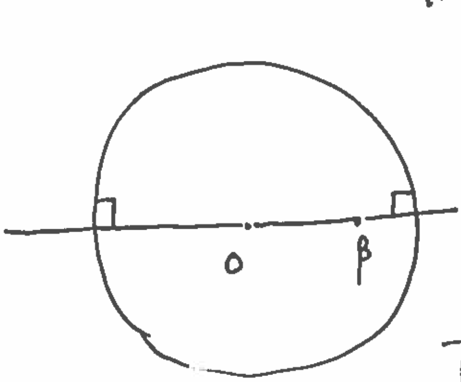
(2)



$\mu_{z_0}(a,b)$ = angle between $\gamma_a'(0)$ and $\gamma_b'(0)$

where $\gamma_a,\gamma_b : [0,1)\to\mathbb{D}$ geodesics ; $\gamma_a(0)=\gamma_b(0)=z_0$
$\gamma_a(t),\gamma_b(t)\to a$ as $t\to 1$.

<u>Proof.</u> – (1) Assume $\alpha=0$ and $\beta\in(0,1)$.
If $\gamma:[0,1]\to\mathbb{D}$ is a path joining 0 and β then $u=\operatorname{Re}\gamma$ is another such path and



$$\int_0^1 \frac{|\gamma'(t)|\,dt}{1-|\gamma(t)|^2} \geq \int_0^1 \frac{|u'(t)|\,dt}{1-|u(t)|^2}$$

This shows that $\mathbb{R}$ is the unique geodesic through 0 and $\beta\in(0,1)$.

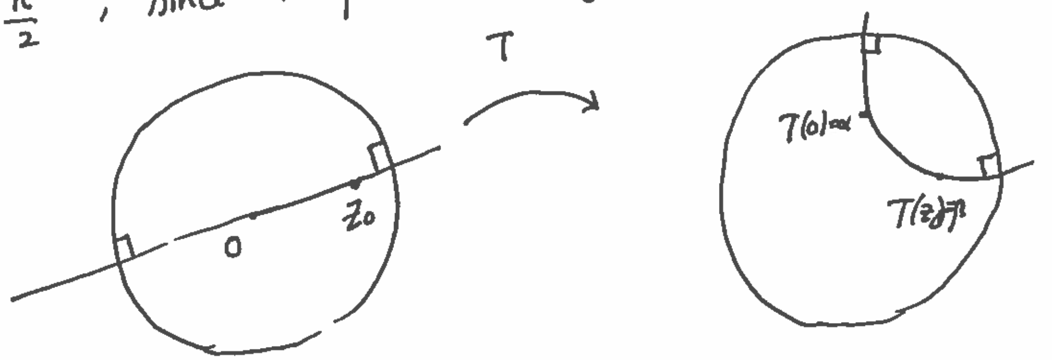
Using rotations, we conclude the same for $\alpha = 0$ ⑦

$\beta \in \mathbb{D} \setminus \{0\}$ arbitry : $\mathbb{R}\cdot\beta$ is the unique geodesic.

In general, let $T \in Aut(\mathbb{D})$ be such that $T(0) = \alpha$

$T(z_0) = \beta \quad (z_0 \in \mathbb{D}^\times)$

T maps line through $0, z_0$ to circle through α, β - meeting $\partial\mathbb{D}$ at $\frac{\pi}{2}$, since T preserves angles.



□

(2) Obvious when $z_0 = 0$, μ_{z_0} = Lebesgue measure

General case follows from the properties of Möbius transformations and invariance of Poisson kernel. □