Lecture 35

Recall - (1) Riemann Mapping Theorem.- Let $\Omega \subsetneq \mathbb{C}$ be a proper, open, connected and simply connected subset of $\mathbb{C}$. Then there exists a biholomorphism $f : \mathbb{D} \xrightarrow{\sim} \Omega$.

(2) Given a continuous function $g : \partial\mathbb{D} \to \mathbb{R}$,

$$u(z) := \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{i\phi} - z|^2} \, g(e^{i\phi}) \, d\phi$$

is the unique harmonic function $\mathbb{D} \xrightarrow{u} \mathbb{R}$ s.t. u extends to $\overline{\mathbb{D}}$ as cnts. function and $u\Big|_{\partial\mathbb{D}} = g$.

In order to solve the boundary value problem for arbitrary domains, we need to show that (the) Riemann map extends to the boundary. This result is due to Caratheodory - Osgood (independently) around 1910's. We will discuss a somewhat simpler version of this result in this lecture.

§1. Definition. - Let $\mathcal{U} \subset \mathbb{C}$ be an open set and $z_0 \in \partial\mathcal{U} = \overline{\mathcal{U}} \setminus \mathcal{U}$.

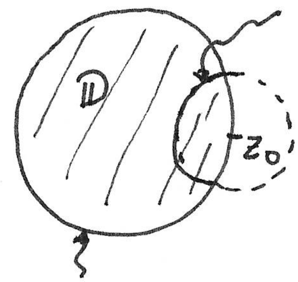
We say z_0 is <u>accessible</u> if $\exists\, r_0 > 0$ s.t. $\forall\, r \in (0, r_0)$

$C(z_0; r) \cap \mathcal{U}$ is an arc - i.e. $\exists\, \theta_1(r) < \theta_2(r)$ s.t.

$$\mathcal{U} \cap C(z_0; r) = \{ z_0 + re^{i\theta} : \theta_1(r) < \theta < \theta_2(r) \}.$$

$$\boxed{\begin{array}{c}\theta_2(r) - \theta_1(r) \leq 2\pi \\ \forall r\end{array}}$$

e.g. (i) $\mathcal{U} = \mathbb{D}$. Every $z_0 \in \partial\mathbb{D}$ is accessible.

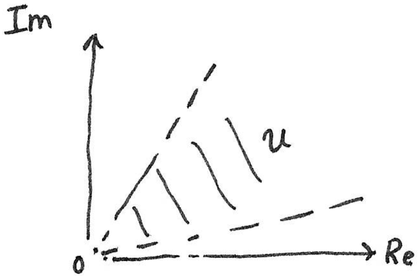


$\mathbb{D} \cap C(z_0; r)$ is an arc of opening $< \pi$.

$C(0;1)$

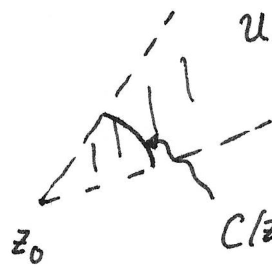
(ii) Corners are accessible.

$\mathcal{U} = \{\phi_1 < \arg(z) < \phi_2\}$



$0 \in \partial\mathcal{U}$ is accessible

(every $z_0 \in \partial\mathcal{U}$ is accessible).



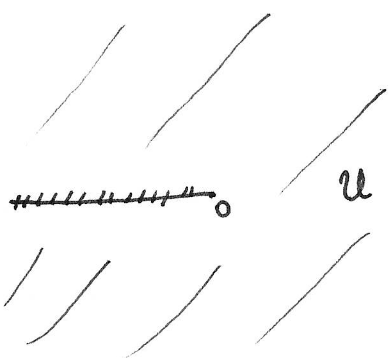
$C(z_0; r) \cap \mathcal{U}$

= arc of opening

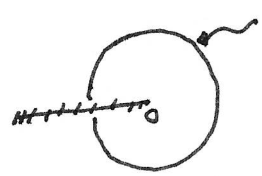
$\phi_2 - \phi_1$.

(iii) Cuts are not accessible

$\mathcal{U} = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. $0 \in \partial\mathcal{U}$ is the only accessible point on $\partial\mathcal{U}$.

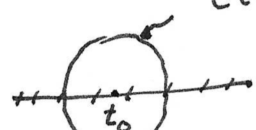


Near 0



$C(0; r) \cap \mathcal{U}$ is an arc of opening 2π.

$t_0 \in \mathbb{R}_{<0}$



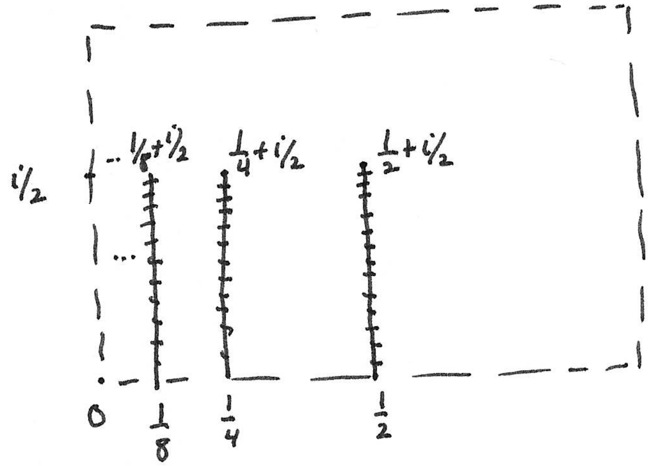
$C(t_0; r) \cap \mathcal{U}$ is not an arc. $(r < |t_0|)$

Rk. - The same argument will show that in this case the Riemann map has 2 limits $\lim\limits_{\substack{z \to t_0 \\ \mathrm{Im}(z) > 0}} f(z) \neq \lim\limits_{\substack{z \to t_0 \\ \mathrm{Im}(z) < 0}} f(z)$.

(iv) If C is any contour in $\mathbb{C}$ and $\mathcal{U} = \text{Interior}(C)$ then every point $z_0 \in C$ is accessible. (Exercise).

(v) $$\mathcal{U} = \{z \in \mathbb{C} : 0 < \mathrm{Re}(z), \mathrm{Im}(z) < 1\} \setminus \bigcup_{n=1}^{\infty} \left\{\frac{1}{2^n} + it : 0 < t \leq \frac{1}{2}\right\}$$

$[0, \frac{i}{2}] \subset \partial\mathcal{U}$ are all inaccessible points.



§2. Caratheodory / Osgood-Taylor extention theorem.

> Assume $f : \Omega \to \mathbb{D}$ is a conformal equivalence and $z_0 \in \partial\Omega$ is an accessible point. Then $\displaystyle\lim_{\substack{z \to z_0 \\ z \in \Omega}} f(z)$ exists.

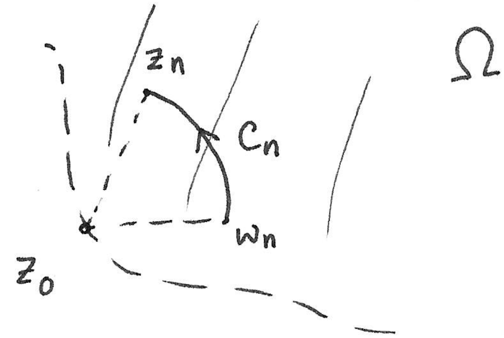
Proof.- Assuming the contrary, we can find two sequences $\{z_n\}$, $\{w_n\}$ in Ω so that $r_n = |z_n - z_0| = |w_n - z_0| \to 0$ as $n \to \infty$

$$\left| \lim_{n\to\infty} f(z_n) - \lim_{n\to\infty} f(w_n) \right| \neq 0.$$

[Easy exercise - such sequences can always be found]

By accessibility assumption, for $n \gg 0$,

$C(z_0; r_n) \cap \Omega$ is an arc - hence z_n & w_n can be joined by an arc.



So, $d_n = |f(z_n) - f(w_n)|$

$$= \left| \int_{C_n} f'(z)\, dz. \right| .$$

$$C_n = \{ z_0 + r_n e^{i\theta} : \theta_1 \leq \theta \leq \theta_2 \}$$

By assumption on limits, for $n \gg 0$, $d_n > d > 0$ for some d.

$$\Rightarrow \quad d^2 < d_n^2 = \left| \int_{C_n} f'(z) dz \right|^2 = r_n^2 \left| \int_{\theta_1}^{\theta_2} f'(z_0 + r_n e^{i\theta}) e^{i\theta} d\theta \right|^2$$

$$\leq r_n^2 \int_{\theta_1}^{\theta_2} |f'(z_0 + r_n e^{i\theta})|^2 d\theta \cdot \underbrace{\int_{\theta_1}^{\theta_2} |e^{i\theta}|^2 d\theta}_{\theta_2 - \theta_1 \leq 2\pi.}$$

$$\frac{d^2}{r_n} \leq 2\pi \int_{\theta_1}^{\theta_2} |f'(z_0 + r_n e^{i\theta})|^2 r_n \, d\theta$$

$$\Rightarrow \quad \underbrace{\text{Area}(f(\Omega \cap D(z_0; r)))}_{\leq \text{Area}(\mathbb{D}) = \pi.} \geq \int_0^r \frac{d^2}{2\pi r} dr = +\infty$$

Contradiction. $\square$

§3. Corollary of Theorem §2.

Let C_1, C_2 be two contours in $\mathbb{C}$, $\Omega_j = \text{Interior}(C_j)$ $j = 1,2$, and $f : \Omega_1 \xrightarrow{\sim} \Omega_2$ a conformal equivalence. Then f extends to a homeomorphism $\overline{\Omega}_1 \to \overline{\Omega}_2$.

Remark. - In some important special cases - one could use Schwarz' reflection principle to extend a Riemann map across a boundary part. Clearly - if it can be done - the corresponding boundary part must be analytic. This is the content of a theorem due to Schwarz. We will discuss below, the cases when boundary part is a line or a circle.

§4. Schwarz' reflection principle. - version 1.

Let $D \subset \mathbb{H}$ be a open subset of the upper half plane so that $\overline{D} \cap \mathbb{R}$ contain an non-empty interval I $(I \subset \overline{D} \cap \mathbb{R} = \partial D \cap \mathbb{R})$.

Let $f : D \to \mathbb{C}$ be a holomorphic function that extends to a cnts function on $D \cup I$, s.t. $f(x) \in \mathbb{R}$ $\forall\, x \in I$.

Then,
$$\tilde{f}(z) := \begin{cases} f(z) & , z \in D \\ \lim\limits_{\substack{w \to z \\ w \in D}} f(w) & , z \in I \subset \partial D \cap \mathbb{R} \\ \overline{f(\overline{z})} & , z \in D^* = \rho_{\mathbb{R}}(D) = \{\overline{w} : w \in D\} \end{cases}$$

is holomorphic on $D \cup I \cup D^*$.

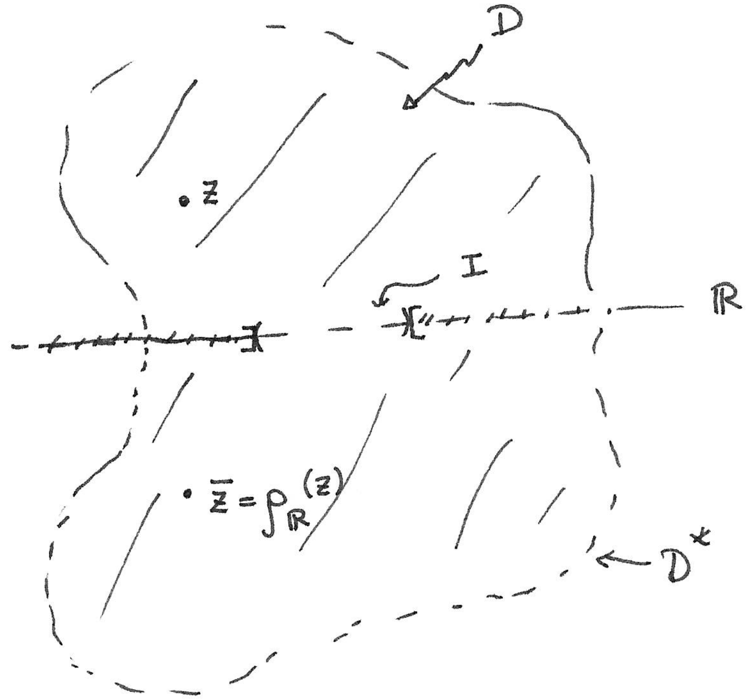
Proof.- By Morera's theorem, it is enough to show that

$$\int_{\partial R} \tilde{f}(z)\,dz = 0 \quad -(*)$$

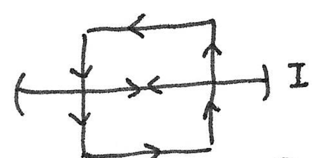
(just write f instead of $\tilde{f}$).

for every rectangle $R \subset \Omega$

$(\Omega = D \cup I \cup D^*)$.



Note - if R is entirely in D or D^* we are done - since f and $z \mapsto \overline{f(\bar{z})}$ are holomorphic. By cutting

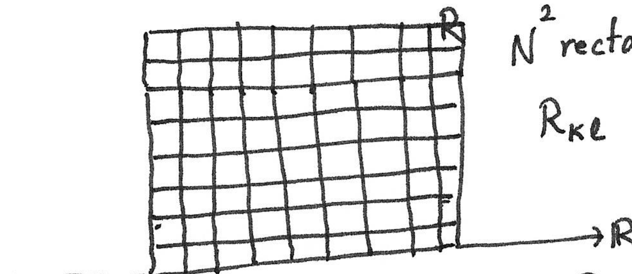


, we may assume that the base of R is supported on the interval I. Let P = perimeter of R.

Given $\varepsilon > 0$, pick $N \gg 0$ s.t.

$$|f(z) - f(z')| < \varepsilon, \ \forall\ |z - z'| < \frac{d}{N}.$$

$(z, z' \in D \cup I)$



N^2 rectangles $R_{k\ell}$

perimeter of $R_{k,\ell} = \dfrac{P}{N}$.

(say $R_{k,N}$ are based on $\mathbb{R}$, $k = 1, \dots, N$)

$$\left| \int_{\partial R} f(z)\,dz \right| \leq \sum_{k,\ell} \left| \int_{\partial R_{k,\ell}} f(z)\,dz \right|$$

$$= \sum_{k=1}^{N} \left| \int_{\partial R_{kN}} f(z)\,dz \right| \qquad (\text{for } \ell \neq N,\ R_{k,\ell} \subset D)$$

$$\# = \sum_{K=1}^{N} \left| \int_{\partial R_{KN}} (f(z) - f(z_K))\, dz \right| \qquad z_K \in \text{interior}(R_{KN})$$

$$\leq \sum_{K=1}^{N} \varepsilon \cdot \frac{P}{N} = \varepsilon \cdot P$$

$$\Rightarrow \left| \int_{\partial R} f(z)\, dz \right| < \varepsilon' \text{ for every } \varepsilon' > 0 \text{, hence, it must be zero.} \qquad \square$$

§5. Remarks.- (1) Combining the reflection principle - version 1 - (for $\rho_{\mathbb{R}}$) and identity theorem, we conclude that:

[if $D \subset \mathbb{C}$ (open, connected) is s.t. $D \cap \mathbb{R}$ contains an interval, and $f : D \to \mathbb{C}$ is hol. s.t. $f(x) \in \mathbb{R}$ $\forall x \in D \cap \mathbb{R}$, then $f(\bar{z}) = \overline{f(z)}$.]

(2) Using Möbius transformations and the fact that they ~~preserve~~ commute with reflections, we have similar result for boundary parts which are circular arcs - Exercise - State this precisely.