Lecture 36

Recall : Last time we studied boundary behaviour of conformal equivalences:

$$f : \Omega \xrightarrow{\sim} \mathbb{D} \; , \; z_0 \in \partial\Omega \text{ accessible } \Rightarrow \lim_{\substack{z \to z_0 \\ z \in \Omega}} f(z) \text{ exists.}$$

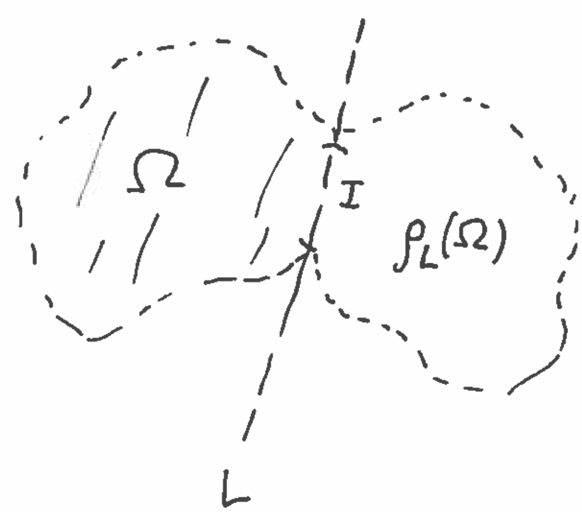
(Caratheodory ; Osgood - Taylor)

Thus, a conformal equivalence can be extended to the boundary, for domains Ω bounded by a contour.

Assuming part of the boundary is a line or a circular arc. Schwarz' reflection principle allows to extend f across that part.

(Another formulation - slightly different from last time)

Let $f : \Omega \to \mathbb{H}$ be a conformal equivalence. Assume there is a line L in $\mathbb{C}$ s.t. $\partial\Omega \cap L$ contains a segment I.



Then

$$\tilde{f} : \Omega \cup I \cup \rho_L(\Omega) \to \mathbb{C}$$

$$\tilde{f}(\rho_L(z)) = \overline{f(z)}$$

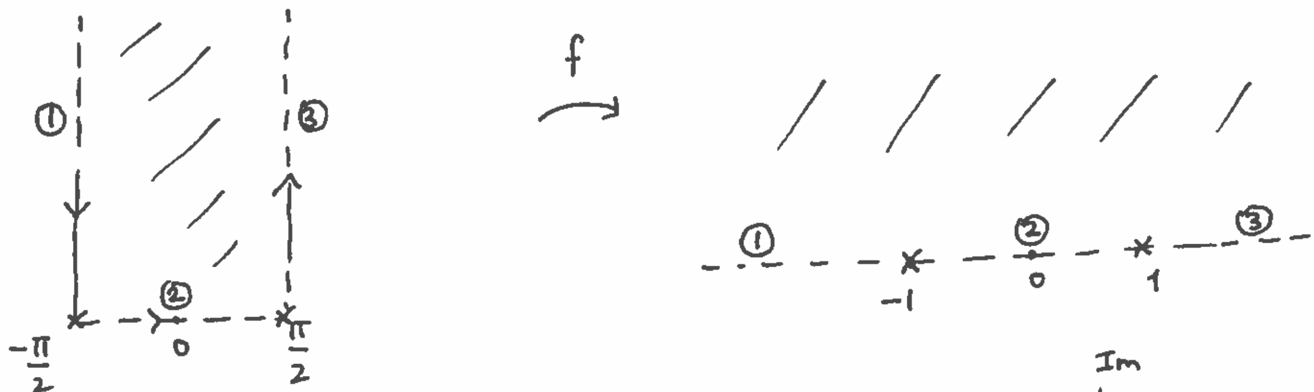
is again holomorphic.

§1. Example.

$$\Omega = \left\{ z \in \mathbb{C} : -\tfrac{\pi}{2} < \mathrm{Re}(z) < \tfrac{\pi}{2} \ , \text{ and } \mathrm{Im}(z) > 0 \right\}$$

$f : \Omega \to \mathbb{H}$ uniquely determined by $f(\pm \frac{\pi}{2}) = \pm 1$, $f(0) = 0$.

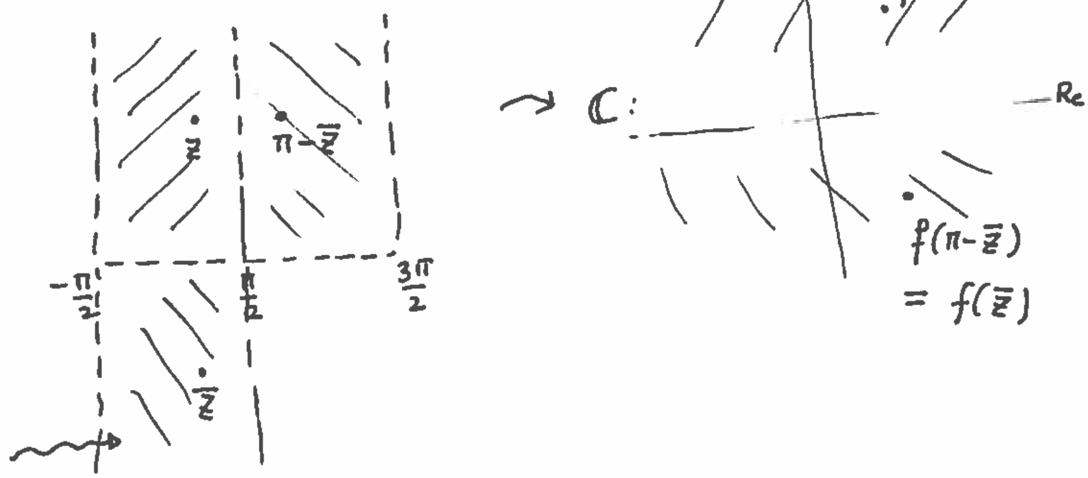
(recall: $f(z) = \sin(z)$).



3 boundary components

Reflection in 3 :

Reflection in 2



$\Rightarrow \quad f(z) = f(\pi - z)$ Periodicity of sine function.

Note: $\quad \sin^{-1}(w) = \displaystyle\int_0^w \frac{ds}{\sqrt{1-s^2}} \ : \ \mathbb{H} \longrightarrow$

i.e. $\quad \dfrac{d}{dw}\left(\sin^{-1}(w)\right) = \dfrac{1}{\sqrt{1-w^2}} \ , \quad \sin^{-1}(0) = 0.$

Note - due to the multi-valued nature of $(1-w^2)^{\frac{1}{2}}$ - it is more natural to further differentiate it:

$$g(w) = (1-w^2)^{-\frac{1}{2}} = (1-w)^{-\frac{1}{2}}(1+w)^{-\frac{1}{2}} \quad \text{solves}$$

$$\frac{g'(w)}{g(w)} = \frac{-1/2}{1+w} + \frac{+1/2}{1-w} \quad ; \quad g'(0) = 1.$$

Hence, arc-sine or $\sin^{-1}$ can be defined as (the) function solving

$$\frac{f''(w)}{f'(w)} = \frac{-1/2}{1+w} + \frac{+1/2}{1-w} \quad ; \quad f'(0) = 1, \quad f(0) = 0.$$

Schwarz-Christoffel formula generalizes this to arbitrary <u>polygonal region</u> (i.e., regions bounded by lines).

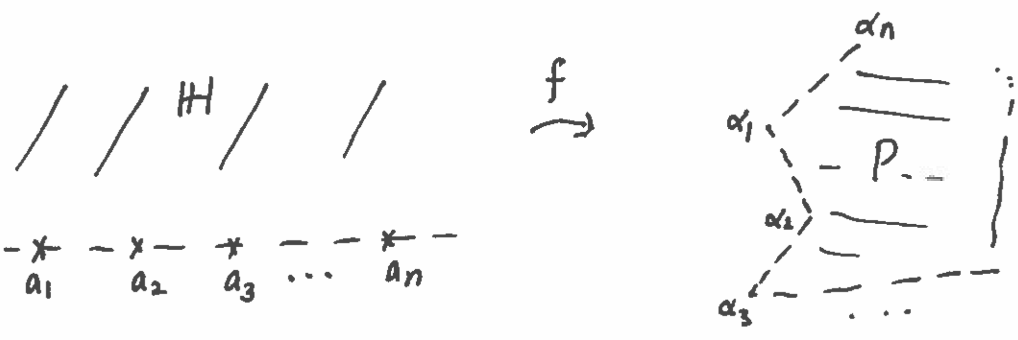
§2. <u>Schwarz-Christoffel formula</u>:

Let $f : \mathbb{H} \to P$ be a conformal equivalence :

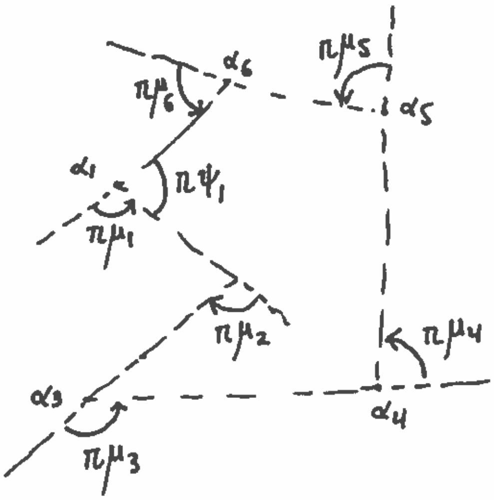
$a_1 < a_2 < \cdots < a_n$ n- distinct points on $\mathbb{R} = \partial \mathbb{H}$

P = polygonal region with vertices

$\alpha_1 = f(a_1), \ldots, \alpha_n = f(a_n)$.



Then, $$\boxed{\frac{f''(z)}{f'(z)} = -\sum_{j=1}^{n} \frac{\mu_j}{z-a_j}}$$ where $\pi\mu_j$ is the "external angle" at j-th vertex of P.



Note: $-1 < \mu_j < 1$

$\mu_1 + \dots + \mu_n = 2$
(total rotation $= 2\pi$).

Let $\psi_j = 1 - \mu_j \in (0,2)$.

(μ_2 is negative – clockwise rotation)

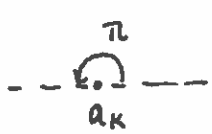
<u>Proof.</u> – Note – f extends to $\overline{\mathbb{H}} \to \overline{P}$ (closed polygon) as a continuous function; and can be analytically continued across any segment (a_j, a_{j+1}) $0 \leq j \leq n$; $a_0 = -\infty$, $a_{n+1} = +\infty$.

f is regular at ∞ ($f(\infty) \in$ segment joining α_n to α_1).

Near a_k : $\quad f(z) = (z-a_k)^{\psi_k} \cdot \underbrace{h_k(z)}_{} + \alpha_k$
$(1 \leq k \leq n)$

holomorphic near a_k.
$h_k(a_k) \neq 0$.

(reason : consider $z \mapsto (f(z)-\alpha_k)^{1/\psi_k}$. This function maps



to

, hence extends across a_k by ar

Schwarz' reflection principle. So, $h(z) = (f(z) - \alpha_K)^{1/\psi_K}$

is holomorphic near a_K and $h(a_K) = 0$.

$\Rightarrow \quad (f(z) - \alpha_K)^{1/\psi_K} = (z - a_K) \cdot h_1(z)$ as claimed $\square$)

$$\Rightarrow \quad \frac{f''(z)}{f'(z)} = \frac{\psi_K - 1}{z - a_K} + \underbrace{\rho_K(z)}_{\text{holomorphic near } a_K.} \quad \text{near } z = a_K$$

Hence $\displaystyle \frac{f''(z)}{f'(z)} - \sum_{K=1}^{n} \frac{\psi_K - 1}{z - a_K}$ is an entire function,

vanishing at ∞ (as f is regular at ∞, $\underset{\text{near } \infty}{f(z) = f(\infty) + c_1 z^{-1} + c_2 z^{-2} + \cdots}$

$\Rightarrow \quad f'(z) \sim z^{-2}$, $f''(z) \sim z^{-3}$ $\quad \Rightarrow \quad f''/f' \to 0$ as $z \to \infty$.)

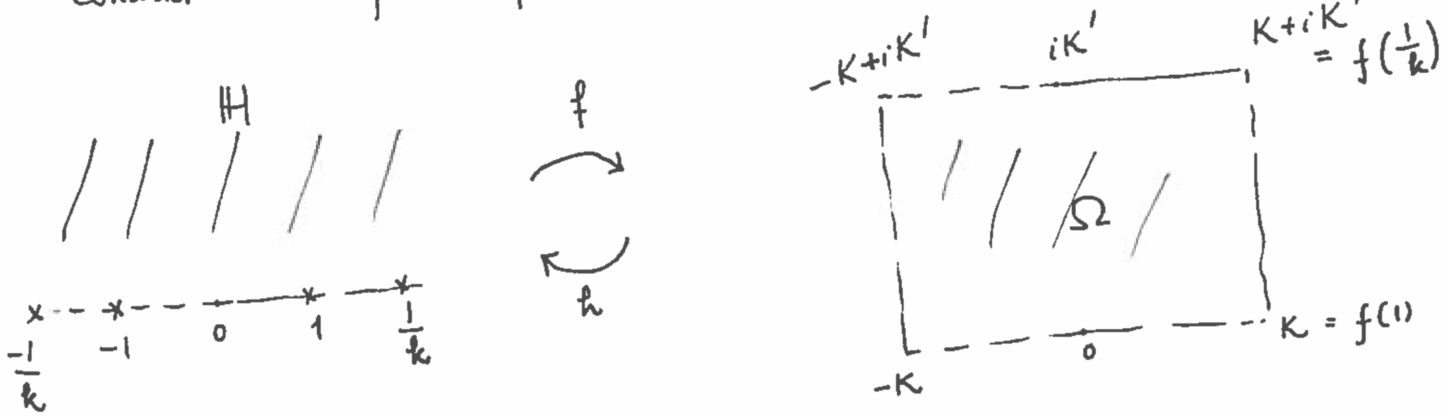
By Liouville's Theorem, $\displaystyle \frac{f''(z)}{f'(z)} - \sum_{K=1}^{n} \frac{\psi_K - 1}{z - a_K} = 0$ $\quad (\mu_K = 1 - \psi_K)$.

$\square$

Explicitly, $$f(z) = A \int_{z_0}^{z} \frac{dw}{\prod_{j=1}^{n} (w - a_j)^{\mu_j}} + B .$$

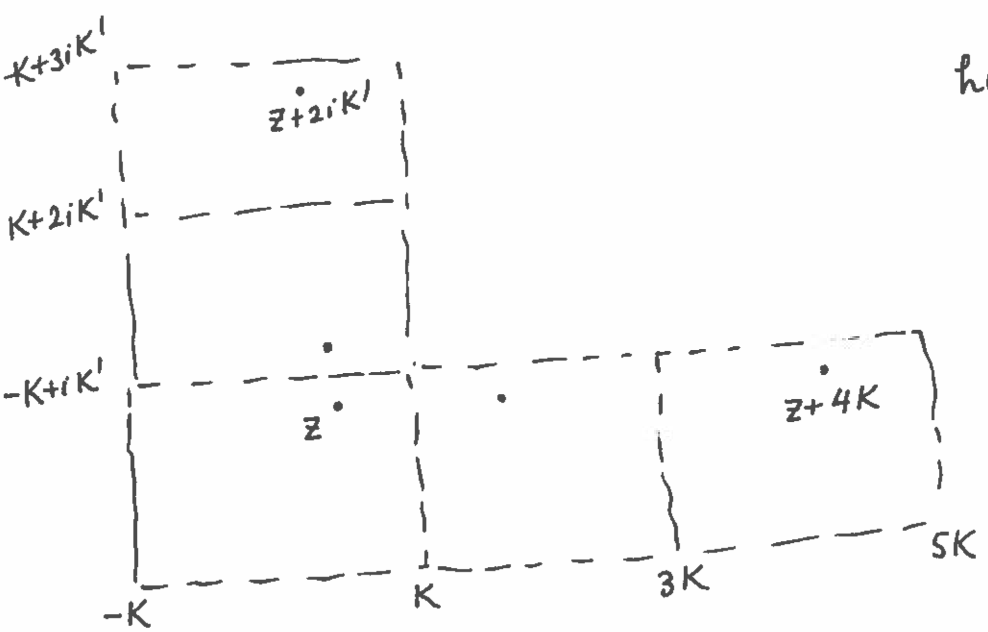
§3. Example - elliptic functions and elliptic integrals.

Consider the conformal equivalence



$k \in (0, 1)$

By reflection principle $4K$ and $2iK'$ are periods of h



$$h(z+4K) = h(z) = h(z+2iK')$$

h is the Jacobian sine function denoted by

$$h(z) = h(z; k)$$

Schwarz - Christoffel formula gives

$$f(z) = \int_0^{z_0} \frac{dt}{\sqrt{1-t^2}\,\sqrt{1-k^2t^2}} \qquad : \text{ elliptic integral}$$

Note :

$$K = \int_0^1 \frac{dt}{\sqrt{1-t^2}\,\sqrt{1-k^2t^2}} = \frac{\pi}{2} \quad \text{at } k=0.$$

$$i\,K' = \int_1^{1/k} \frac{dt}{\sqrt{1-t^2}\,\sqrt{1-k^2t^2}}$$