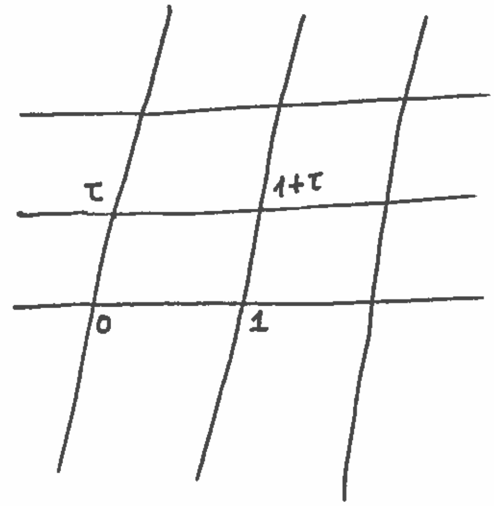
Lecture 37

Notations.- Let $\tau \in \mathbb{C}$, $\mathrm{Im}(\tau) > 0$.

$$\Lambda_\tau := \mathbb{Z} + \mathbb{Z}\tau \subset \mathbb{C}$$



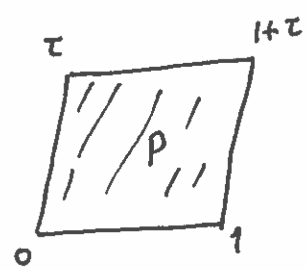
Lattice $\Lambda_\tau \subset \mathbb{C}$

A <u>doubly periodic</u>, or <u>elliptic</u> function is a meromorphic function $f : \mathbb{C} \dashrightarrow \mathbb{C}$ such that

$$f(z+1) = f(z) = f(z+\tau).$$

§1. <u>Prop.</u>- If f is holomorphic and doubly-periodic, then f is constant.

<u>Proof</u>.- Let $P = \{ s + t\tau : 0 \leq s,t \leq 1 \}$



As P is compact and $z \mapsto |f(z)|$ is continuous, there exists $M \in \mathbb{R}_{>0}$ s.t.

$$|f(z)| \leq M \quad \forall z \in P.$$

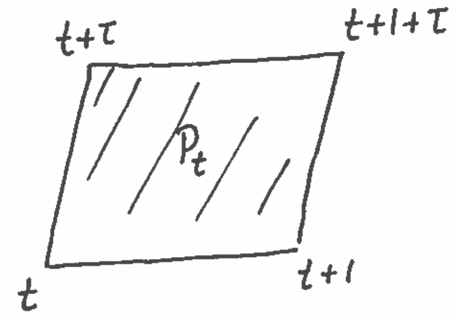
By double-periodicity, we conclude that $|f(z)| \leq M \quad \forall z \in \mathbb{C}$ - i.e., f is a bounded entire function, hence a constant by Liouville's theorem. $\square$

§2. Now, assume that $f : \mathbb{C} \dashrightarrow \mathbb{C}$ is a doubly-periodic, non-constant function. Note - the sets of zeroes and poles of f are discrete subsets of $\mathbb{C}$. Hence, for any $t \in \mathbb{C}$,

$\left(\text{Zeroes of } f\right) \cap P_t$
$\left(\text{Poles of } f\right) \cap P_t$ are finite sets, where

$$P_t = \{t + s_1 + s_2\tau \mid 0 \leq s_1, s_2 \leq 1\}.$$

Assume $t \in \mathbb{C}$ is such that f has no zeroes or poles on $C_t = \partial P_t$.



Let $\alpha_1, \dots, \alpha_m \in P_t$ be zeroes of f (listed according to their multiplicity)
$\beta_1, \dots, \beta_n \in P_t$ be poles of f.

§3. Theorem. – (i) $\displaystyle\sum_{j=1}^{n} \operatorname{Res}_{z=\beta_j} (f(z)) = 0.$

(ii) $m = n$.

(iii) $\displaystyle\sum_{j=1}^{n} \alpha_j - \sum_{j=1}^{n} \beta_j \in \Lambda_\tau$.

Proof. – (i) $$\sum_{j=1}^{n} \operatorname{Res}_{\beta_j}(f) = \frac{1}{2\pi i}\int_{C_t} f(z)\,dz$$

$$= \frac{1}{2\pi i}\left(\int_t^{t+1} + \int_{t+1}^{t+1+\tau} + \int_{t+1+\tau}^{t+\tau} + \int_{t+\tau}^{t} f(z)\,dz\right)$$

$$= \frac{1}{2\pi i}\left(\left(\int_t^{t+1} + \int_t^{t+\tau} + \int_{t+1}^{t} + \int_{t+\tau}^{t} f(z)\,dz\right)\right)$$

by $f(z+1) = f(z) = f(z+\tau)$

$$= 0.$$

(ii) Repeat the same argument as above for $\dfrac{f'(z)}{f(z)}$

$$m - n = \frac{1}{2\pi i}\int_{C_t} \frac{f'(z)}{f(z)}\,dz = 0.$$

(iii) $$\sum_{j=1}^{n} \alpha_j - \sum_{j=1}^{n} \beta_j = \frac{1}{2\pi i}\int_{C_t} z\cdot\frac{f'(z)}{f(z)}\,dz$$

Now, $$\frac{1}{2\pi i}\int_{C_t} z\cdot\frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_t^{t+1}\left(z\frac{f'(z)}{f(z)} - (z+\tau)\frac{f'(z)}{f(z)}\right)dz$$

$$+ \frac{1}{2\pi i}\int_t^{t+\tau}\left(-z\frac{f'(z)}{f(z)} + (z+1)\frac{f'(z)}{f(z)}\right)dz$$

$=$ Winding number of $\{f(t+s\tau) : 0 \leq s \leq 1\}$ around 0

$-\tau\cdot$ (Winding number of $\{f(t+s) : 0 \leq s \leq 1\}$ around 0)

$\in \mathbb{Z} + \tau\mathbb{Z}.$ $\square$

§4. Jacobi's theta function $\theta(z;\tau)$ is the unique holomorphic function of $z \in \mathbb{C}$ satisfying the following properties:

(Periodicity) $\theta(z+1;\tau) = -\theta(z;\tau)$

$$\theta(z+\tau;\tau) = -e^{-\pi i\tau} e^{-2\pi i z}\, \theta(z;\tau)$$

(Zeroes) $\theta(z;\tau) = 0 \iff z \in \Lambda_\tau$.

Normalization: $\theta'(0;\tau) = 1$.

Remark.- The uniqueness of $\theta(z;\tau)$ follows from Prop. §1 above.

Before proving its existence, let us record the converse to Theorem §3 using $\theta(z;\tau)$.

§5. Lemma.- Let $a_1, \dots, a_n; b_1, \dots, b_n \in \mathbb{C}$ be such that

$$\sum_{j=1}^n a_j = \sum_{j=1}^n b_j .$$

(Note: $a_1, \dots, a_n$ need not be distinct (similarly, $b_1, \dots, b_n$ need not be distinct))

Then $F(z) = \prod_{j=1}^n \dfrac{\theta(z-a_j;\tau)}{\theta(z-b_j;\tau)}$ is doubly-periodic.

Conversely, every doubly-periodic function is of this form.

Proof - Let us drop τ from $\theta(z;\tau)$ - for notational convenience.

We first verify that $F(z)$ is doubly-periodic:

$$F(z+1) = \prod_{j=1}^{n} \frac{\theta(z+1-a_j)}{\theta(z+1-b_j)} = \prod_{j=1}^{n} \frac{-\theta(z-a_j)}{-\theta(z-b_j)} = F(z).$$

$$F(z+\tau) = \prod_{j=1}^{n} \frac{-e^{-\pi i\tau}\, e^{-2\pi i(z-a_j)}}{-e^{-\pi i\tau}\, e^{-2\pi i(z-b_j)}} \frac{\theta(z-a_j)}{\theta(z-b_j)}$$

$$= e^{2\pi i\left(\sum a_j - \sum b_j\right)} \cdot F(z) = F(z).$$

For the converse - let $f : \mathbb{C} \dashrightarrow \mathbb{C}$ be a doubly-periodic function. Let $\begin{matrix}\alpha_1, \dots, \alpha_n \\ \beta_1, \dots, \beta_n\end{matrix}$ be $\begin{matrix}\text{zeroes of } f \\ \text{poles of } f\end{matrix}$ within a fundamental parallelogram (as in §2 above). Since $\sum \alpha_j - \sum \beta_j \in \Lambda_\tau$, we can replace $\begin{matrix}\alpha_j \equiv a_j \\ \beta_j \equiv b_j\end{matrix}$ mod Λ_τ so that $\sum a_j = \sum b_j$.

So, both $f(z)$ and $\displaystyle\prod_{j=1}^{n} \frac{\theta(z-a_j)}{\theta(z-b_j)}$ are doubly-periodic and have the same set of zeroes and poles. Hence, their ratio is a holomorphic, doubly-periodic function which must be a constant by Prop §1 above. $\square$