Lecture 38

Recall: let $\tau \in \mathbb{H}$ (i.e. $\mathrm{Im}(\tau) > 0$). We introduced Jacobi's theta function $\theta(z;\tau)$ as the unique holomorphic function of $z \in \mathbb{C}$ satisfying:

(1) $$\theta(z+1;\tau) = -\theta(z;\tau)$$ (Periodicity)
$$\theta(z+\tau;\tau) = -e^{-\pi i\tau} e^{-2\pi i z}\,\theta(z;\tau)$$

(2) $$\theta(z;\tau) = 0 \iff z \in \mathbb{Z} + \tau\mathbb{Z} = \Lambda_\tau$$ (Zeroes)

(3) $$\theta'(0;\tau) = 1$$ (Normalization)

We also proved that given $a_1, \dots, a_n ; b_1, \dots, b_n \in \mathbb{C}$ (not nec. distinct) such that

$\sum_{j=1}^n a_j = \sum_{j=1}^n b_j$, the function $\prod_{j=1}^n \dfrac{\theta(z-a_j;\tau)}{\theta(z-b_j;\tau)}$ is a doubly-periodic function. Moreover, every doubly-periodic function is of this form.

Multiplicative notation. $q = e^{\pi i\tau}$, note: $|q| < 1$.

$$w = e^{2\pi i z}$$

$z \mapsto z+1$ does not effect w

$z \mapsto z+\tau \longleftrightarrow w \mapsto q^2 w.$

§1. Infinite sum expression.

Prop. - $T(z) := \sum_{n\in\mathbb{Z}} (-1)^n q^{n(n-1)} w^n \quad \left(\begin{array}{l} q = e^{\pi i \tau} \\ w = e^{2\pi i z} \end{array} \right)$

converges uniformly on compact subsets of $\mathbb{C}$, hence defines a holomorphic function of $z \in \mathbb{C}$. This function satisfies the following properties.

(1) $T(z+1) = T(z)$ and $T(z+\tau) = -e^{-2\pi i z} T(z)$

(2) T has simple zeroes at points of $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$.

Proof. - Convergence: Let $K \subset \mathbb{C}$ be a compact set. Choose $A \in \mathbb{R}_{>0}$ s.t. $|\mathrm{Im}(z)| < A$, $\forall z \in K$. Then

$$|e^{2\pi i n z}| = e^{-\mathrm{Im}(z)\cdot 2\pi n} < e^{2\pi |n| A}, \quad \forall \begin{array}{l} n \in \mathbb{Z} \\ z \in K. \end{array}$$

So, the series

$$T(z) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n(n-1)} w^n = \sum_{\ell=0}^{\infty} (-1)^\ell q^{\ell(\ell+1)} \left(w^{-\ell} + w^{\ell+1} \right)$$

is dominated by $\displaystyle 2\sum_{\ell=0}^{\infty} |q|^{\ell(\ell+1)} \left(e^{2\pi A}\right)^{\ell+1}$

which converges by ratio test as $|q| < 1$.

The periodicity properties of $T(z)$ are easy to verify.

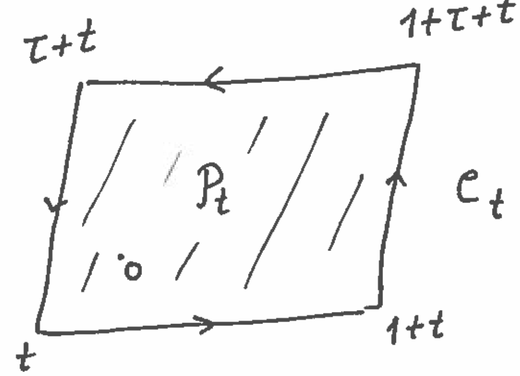
It is also clear from the expression of $T(z)$:

$$T(z) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n(n-1)} w^n$$

$$= \underset{(n=0)}{1} - \underset{(n=1)}{w} + \underset{(n=2)}{q^2 w^2} - \underset{(n=-1)}{q^2 w^{-1}} - \underset{(n=3)}{q^6 w^3} + \underset{(n=-2)}{q^6 w^{-2}} - \dots$$

$$= \sum_{\ell=0}^{\infty} (-1)^\ell q^{\ell(\ell+1)} (w^{-\ell} - w^{\ell+1}) = 0 \text{ at } w = 1$$

that $T(z) = 0$ for $z = 0$, and hence by periodicity properties of T, $T(z) = 0 \;\; \forall z \in \Lambda_\tau$. We only have to prove that all these zeroes are of multiplicity one.



Let $t \in \mathbb{C}$ be s.t. $T(z)$ has no zeroes on $\mathcal{C}_t$, and $0 \in P_t$

$$\left(\begin{array}{l} P_t = \{ t + s_1 + s_2 \tau : 0 \leq s_1, s_2 \leq 1 \} \\ \mathcal{C}_t = \partial P_t . \end{array} \right)$$

$$\begin{array}{l} \# \text{Zeroes} - \# \text{Poles} \\ \text{of } T(z) \text{ within } \mathcal{C}_t \end{array} = \frac{1}{2\pi i} \int_{\mathcal{C}_t} \frac{T'(z)}{T(z)} \, dz \quad \left(\begin{array}{c} \text{argument} \\ \text{principle} \end{array} \right)$$

$$= \frac{1}{2\pi i} \left\{ \int_t^{t+1} + \int_{t+1}^{t+1+\tau} + \int_{t+1+\tau}^{t+\tau} + \int_{t+\tau}^{t} \frac{T'(z)}{T(z)}\, dz \right\}$$

cancel since T is 1-periodic.

Moreover $T(z+\tau) = -e^{-2\pi i z}\, T(z)$

$$\Rightarrow \quad \frac{T'(z+\tau)}{T(z+\tau)} = -2\pi i + \frac{T'(z)}{T(z)}.$$

Hence
$$\frac{1}{2\pi i} \int_{C_t} \frac{T'(z)}{T(z)}\, dz = \frac{1}{2\pi i} \int_t^{t+1} \frac{T'(z)}{T(z)} - \frac{T'(z+\tau)}{T(z+\tau)}\, dz$$

$$= \frac{1}{2\pi i} \int_t^{t+1} 2\pi i\, dz = 1.$$

Meaning, T has exactly one zero within $\mathcal{C}_t$. We already know $T(0) = 0$, so, T only has simple zeroes at Λ_τ as claimed. $\square$

§2. Corollary.- There is a constant $c_1(\tau)$ s.t.

$$\theta(z;\tau) = c_1(\tau)\cdot e^{-\pi i z}\, T(z)$$

$$= c_1(\tau)\cdot \sum_{n\in\mathbb{Z}} (-1)^n q^{n(n-1)} e^{(2n-1)\pi i z}.$$

Proof.- $e^{-\pi i z}\,T(z)$ satisfies the first two of three defining properties of $\theta(z;\tau)$. $\square$

§3. <u>Infinite product expression.</u>

<u>Prop.</u>- $$\theta^+(z) := \prod_{n=1}^{\infty}\left(1 - e^{2\pi i n\tau}\, e^{2\pi i z}\right)$$

$$= \prod_{n=1}^{\infty}\left(1 - q^{2n} w\right)$$ converges uniformly (rel. to compact subsets of $\mathbb{C}$)

This function has the following properties:

(1) $\theta^+(z+1) = \theta^+(z)$ and $\theta^+(z+\tau) = \dfrac{\theta^+(z)}{1-q^2 w}$

(Euler) (2) $$\theta^+(z) = \sum_{\ell=0}^{\infty} (-1)^\ell w^\ell \cdot \frac{q^{\ell(\ell+1)}}{(1-q^2)\cdots(1-q^{2\ell})}$$

(3) $\theta^+(z) = 0 \iff z \in \mathbb{Z} + \tau\,\mathbb{Z}_{\leq -1}$

($z = m - n\tau$; $m\in\mathbb{Z}$, $n\in\mathbb{Z}_{\geq 1}$)

<u>Proof.</u>- Convergence: let $p = q^2 = e^{2\pi i \tau}$. Let a compact set $K \subset \mathbb{C}$ be given. Let $A \in \mathbb{R}$ be s.t. $A < \mathrm{Im}(z)$ $\forall z \in K$.

Choose $N_0 \in \mathbb{N}$ s.t. $|p|^N < \frac{1}{2} e^{2\pi A}$, $\forall N \geq N_0$

(exists, since $|p|^n \to 0$ as $n \to \infty$, $|p| < 1$).

So, for every $z \in K$ and $N \geq N_0$, we have:

$$|p^N w| = |p|^N \cdot e^{-2\pi \mathrm{Im}(z)} \leq |p|^N \cdot e^{2\pi A} < \frac{1}{2}.$$

$$\Rightarrow \left| \log(1 - p^N w) \right| = \left| p^N w + \frac{1}{2} p^{2N} w^2 + \frac{1}{3} p^{3N} w^3 + \dots \right|$$

$$\leq |p|^N \cdot |w| \left(1 + |p|^N \cdot |w| + |p|^{2N} |w|^2 + \dots \right)$$

$$\leq |p|^N \cdot e^{-2\pi A} \left(1 + \frac{1}{2} + \frac{1}{4} + \dots \right) = 2 \cdot e^{-2\pi A} |p|^N .$$

Therefore
$$\left| \sum_{N=N_0}^{\infty} \log(1 - p^N w) \right| \leq 2 \cdot e^{-2\pi A} \sum_{N=N_0}^{\infty} |p|^N < \infty .$$

So,
$$\prod_{N=1}^{\infty} (1 - p^N w) = \prod_{n=1}^{N_0 - 1} 1 - p^n w \cdot \exp\left(\sum_{N=N_0}^{\infty} \log(1 - p^N w) \right)$$

converges uniformly on K. $\square$

Properties (1) and (3) are obvious.

<u>Proof of (2)</u>: Solve $f(pw) = \dfrac{f(w)}{1 - pw}$ near $w = 0$

$f(0) = 1$.

If $f(w) = \sum_{n=0}^{\infty} c_n w^n$; $c_0 = 1$, then the equation

$f(w) = (1-pw) f(pw)$ becomes :

$$c_N = p^N c_N - p^N c_{N-1} \qquad \forall\, N \geq 1 .$$

i.e. $$c_N = \frac{-p^N}{1-p^N} \cdot c_{N-1} = \frac{(-1)(-1)\, p^{N+(N-1)}}{(1-p^N)(1-p^{N-1})} c_{N-2} \cdots$$

$$= \frac{(-1)^N \, p^{N(N+1)/2}}{\prod_{j=1}^{N} (1-p^j)} .$$

□

§4. Corollary.- There is a constant $C_2(\tau)$ such that

$$\begin{aligned} \theta(z) &= C_2(\tau) \sin(\pi z)\, \theta^+(z) \cdot \theta^+(-z) \\ &= C_2(\tau) \sin(\pi z) \prod_{n=1}^{\infty} (1 - q^{2n} w)(1 - q^{2n} w^{-1}) \end{aligned}$$

Proof.- $\sin(\pi z)\, \theta^+(z)\, \theta^+(-z)$ has properties (1) and (2) of the definition of $\theta(z)$.

□

§5. Jacobi's triple product identity.-

$$\boxed{\sum_{n\in\mathbb{Z}} (-1)^n q^{n(n-1)} w^n = \prod_{l=1}^{\infty} (1-q^{2l})(1-q^{2l-2}w)(1-q^{2l}w^{-1})}$$

So,
$$\theta(z;\tau) = \frac{\sin(\pi z)}{\pi} \prod_{n=1}^{\infty} \frac{(1-q^{2n}w)(1-q^{2n}w^{-1})}{(1-q^{2n})^2}$$

$$= \frac{-1}{2\pi i \prod_{n=1}^{\infty}(1-q^{2n})^3} \cdot \sum_{n\in\mathbb{Z}} (-1)^n q^{n(n-1)} e^{2\pi i z(n-\frac{1}{2})}$$

Proof. of Jacobi's triple product identity.- (let $p = q^2$ for notational convenience)

$$P(w;p) := \sum_{n\in\mathbb{Z}} (-1)^n p^{n(n-1)/2} w^n$$

$$Q(w;p) := \prod_{l=1}^{\infty} (1-p^{l-1}w)(1-p^l)(1-p^l w^{-1})$$

satisfy the same functional equation $F(pw) = -w^{-1} F(w)$

By general principles $Q(w;p) = \underbrace{C(p)}_{\text{constant - independent of } w} \cdot P(w;p)$

$$\lim_{p\to 0} P(w,p) = 1 - w = \lim_{p\to 0} Q(w,p)$$

implies that $\quad C(0) = 1.$

A quick verification (left as an easy exercise) shows that

$$P(-p^{\frac{1}{2}}i\,,\,p) = P(p^2, p^4) \qquad (\neq 0)$$

$$Q(-p^{\frac{1}{2}}i\,,\,p) = Q(p^2, p^4).$$ Hence we conclude

$$C(p) = C(p^4) = C(p^{16}) = \dots \to \underset{=1.}{C(0)} \text{ as } |p|<1. \qquad \square$$

§6. Remarks.- (1) Set $p = x^3$ and $w = x$ to get

$$\prod_{n=1}^{\infty} (1-x^{3n-2})(1-x^{3n-1})(1-x^{3n}) = \sum_{m\in\mathbb{Z}} (-1)^m x^{m+\frac{3m^2-3m}{2}}$$

i.e. $$\prod_{n=1}^{\infty}(1-x^n) = \sum_{m\in\mathbb{Z}} (-1)^m x^{(3m-1)m/2}$$

$$\boxed{\prod_{n=1}^{\infty}(1-x^n) = 1 + \sum_{\ell=1}^{\infty} (-1)^\ell \left(x^{(3\ell-1)\ell/2} + x^{(3\ell+1)\ell/2}\right)}$$

Euler's pentagonal number identity.

(2) Other commonly used version of theta function

$$\sum_{m\in\mathbb{Z}} q^{m^2} w^m = \prod_{n=1}^{\infty} (1+q^{2n-1}w)(1+q^{2n-1}w^{-1})(1-q^{2n})$$

Functional equation $\quad F(q^2 w) = q^{-1} w^{-1} F(w)$.

Zeroes at $\quad \frac{1+\tau}{2} + \Lambda_\tau$.

(3) Fourier's derivation - Heat equation (1822)

Solve for $\psi(x,t)$ such that $\quad \partial_t \psi = A\, \partial_x^2 \psi \quad$ (A: constant)

$$\psi(x+1,t) = \psi(x,t) \qquad \text{(periodic boundary cond}^{n}\text{)}$$

Sol: If $\quad \psi(x,t) = \sum_{n\in\mathbb{Z}} c_n(t)\, e^{2\pi i n x}$, then

$$c_n'(t) = -4\pi A n^2 c_n(t) \qquad \forall\, n \in \mathbb{Z}$$

$$\Rightarrow \quad c_n(t) = c_n(0)\, e^{-4\pi A n^2 t}$$

If $c_n(0) = 1 \ \forall n$, we get $\quad \vartheta(x,t) := \sum_{n\in\mathbb{Z}} e^{-4\pi A t n^2} e^{2\pi i n x}$.

For arbitrary initial cond^{n} $\psi(x,0) = \phi(x)$, the soln. is given by

$$\psi(x,t) = \int_0^1 \vartheta(x-y,t)\,\phi(y)\,dy.$$

(i.e., theta fn. is the propagator for the heat eq^n.)

(4) Jacobi's inversion formula.

Again, let $\vartheta(x,\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i \tau n^2} e^{2\pi i n x}$. Then,

$$\boxed{\vartheta(x,\tau) = \sqrt{\frac{i}{\tau}}\,\exp\left(\frac{-\pi i x^2}{\tau}\right)\cdot \vartheta\left(\frac{x}{\tau}, \frac{-1}{\tau}\right)}$$

For $x=0$ and $\tau = iA$ $(A \in \mathbb{R}_{>0})$ we get

$$\boxed{\sum_{n\in\mathbb{Z}} e^{-\pi A n^2} = \frac{1}{\sqrt{A}} \sum_{m\in\mathbb{Z}} e^{-\frac{\pi}{A} m^2}}$$

Poisson summation formula.