PROBLEMS IN COMPLEX ANALYSIS

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Problem 1. Let $U \subset \mathbb{C}$ be an open set. Prove that U is connected if, and only if U is path–connected. In this case, show that any two points of U can be joined by a *zig–zag* path, i.e., a path consisting only of horizontal and vertical line segments.

Problem 2. Let $n \in \mathbb{Z}_{\geq 2}$ and $z_1, \ldots, z_n, w_1, \ldots, w_n \in \mathbb{C}$. Prove the following identity:

$$\left| \sum_{k=1}^{n} z_k \overline{w_k} \right|^2 = \left(\sum_{k=1}^{n} |z_k|^2 \right) \left(\sum_{k=1}^{n} |w_k|^2 \right) - \sum_{1 \leq k < \ell \leq n} |z_k w_\ell - z_\ell w_k|^2 \ .$$

This identity is due to Lagrange. It immediately implies the Cauchy–Schwarz inequality.

Problem 3. Let $z_1, z_2, z_3 \in S^1 = \{z : |z| = 1\}$ be such that $z_1 + z_2 + z_3 = 0$. Show that these three points form vertices of an equilateral triangle.

Problem 4. Prove the *Ptolemy relation*: if A, B, C, D are four distinct points on a circle, then

$$|AC||BD| = |AB||CD| + |AD||BC| \ ,$$

where $|PQ|$ denotes the length of the line segment joining P and Q.

Problem 5. Assume $z_1, z_2 \in \mathbb{C}$ are two distinct points. Sketch the following curves as λ varies over $\mathbb{R}_{>0}$ (circles of Apollonius)

$$\{z \in \mathbb{C} : |z - z_1| = \lambda |z - z_2|\} \, .$$

Problem 6. Sketch the following curves for $\lambda \in \mathbb{R}_{>0}$ (Bernoulli's lemniscate is when $\lambda = 1$): $|z^2 - 1| = \lambda$.

Problem 7. Let $g \in C^2(\Omega; \mathbb{R})$, where $\Omega \subset \mathbb{C}$ is an open and connected set. Show that the Laplace equation for g, written in polar coordinates, takes the following form:

$$r^2 \frac{\partial^2 g}{\partial r^2} + r \frac{\partial g}{\partial r} + \frac{\partial^2 g}{\partial \theta^2} = 0 \ .$$

Use this to prove that if g is harmonic, and independent of θ, then $g = A \ln(r) + B$, for some constants $A, B \in \mathbb{R}$.

Problem 8. Let $n \in \mathbb{Z}_{\geq 2}$. Describe the level curves $\mathrm{Re}(z^n) = 0$ and $\mathrm{Im}(z^n) = 0$.

Problem 9. Show that $u(x, y) = \dfrac{x}{x^2 + y^2}$ is harmonic. Compute its harmonic conjugate (the domain is $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$).

Problem 10. Let $a \in [0,1)$ and $r \in [0, 1-a)$. Let $C(a;r)$ be the circle centered at a of radius r, and $S^1 = C(0;1)$ (note: $C(a;r)$ lies inside S^1). Let $A_1 = e^{\iota\psi_1}$ and $A_2 = e^{\iota\psi_2}$ be two points on S^1 such that the line segment A_1A_2 is tangent to $C(a;r)$. What is the relation between ψ_1 and ψ_2? *This calculation is due to Jacobi, who used it to prove Poncelet's porism for circles.*

Problem 11. Let $\Omega \subset \mathbb{C}$ be an open, connected and simply–connected set, $u : \Omega \to \mathbb{R}$ a harmonic function (i.e., $u \in C^2(\Omega, \mathbb{R})$, and $u_{xx} + u_{yy} = 0$). Show that there exists $v : \Omega \to \mathbb{R}$ such that $f = u + \iota v$ is holomorphic. Hence, u is real–analytic. This is an instance of "regularity theorems" in mathematics.

Problem 12. Compute the following integrals.

(1) $\displaystyle\int_C \frac{ze^z}{(z-a)^{n+1}}\,dz$. Here $C = C(a,r)$ is the counterclockwise circle around $a \in \mathbb{C}$ of radius $r \in \mathbb{R}_{>0}$, and $n \in \mathbb{Z}_{\geq 0}$.

(2) $\displaystyle\int_C \frac{\sin(z)}{z(1-z)^3}\,dz$. Here $C = C(0, 1/2)$ is the circle of radius $1/2$ around 0.

Problem 13. Let $p(z) = z^n + \sum_{j=0}^{n-1} a_j z^{n-j} \in \mathbb{C}[z]$ be a polynomial of degree $n \geq 2$. Let $z_1, \dots, z_n \in \mathbb{C}$ be its roots (not necessarily distinct, listed according to their multiplicity). Let $\zeta \in \mathbb{C}$ be a root of $p'(z)$ such that $\zeta \notin \{z_1, \dots, z_n\}$. Show that

$$\left(\sum_{j=1}^{n} \frac{1}{|\zeta - z_j|^2}\right)\zeta = \sum_{j=1}^{n} \frac{z_j}{|\zeta - z_j|^2} \ .$$

This result is called Gauss–Lucas Theorem. It shows that ζ can be written as a linear combination of z_j such that the coefficients are positive real and add up to 1.

Problem 14. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function such that $f(0) = 0$ and $\lim_{z\to\infty} \mathrm{Re}(f(z)) = 0$. Show that $f(z) = 0$ for every $z \in \mathbb{C}$. (Hint: show that e^f is bounded, and use Liouville's theorem).

Problem 15. *Dirichlet's test.* Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of complex numbers. Set $S_k = a_1 + \cdots + a_k$ ($k \geq 1$). Show that:

(1) For every $n, m \in \mathbb{Z}_{\geq 1}$, $n \geq m$, we have:

$$\sum_{k=m}^{n} a_k b_k = \sum_{k=m}^{n-1} S_k(b_k - b_{k-1}) - S_{m-1}b_m + S_n b_m \ .$$

This formula is called *Abel's transformation.*

(2) Assume that there exists $M \in \mathbb{R}_{>0}$ such that $|S_k| \leq M$, for every $k \geq 1$. Further, assume that $\{b_n\}$ is real, monotonically non–increasing, and $\lim_{n\to\infty} b_n = 0$. Show that $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

Problem 16. *Weierstrass' M test.* Let $\{a_n(z)\}_{n=1}^{\infty}$ be a sequence of functions defined on a domain Ω. Assume that, for every compact set $K \subset \Omega$, there exists $M_K(n) \in \mathbb{R}_{>0}$ such that

- $|a_n(z)| < M_K(n)$ for every $z \in K$ and $n \geq 1$.

- $\sum_{n=1}^{\infty} M_k(n)$ is convergent.

Then, $\sum_{n=1}^{\infty} a_n(z)$ converges, uniformly and absolutely, on Ω.

Problem 17. Decide whether the following series are convergent or not. For the second, decide the values of θ for which it converges.

$$\text{(a)} \sum_{n=1}^{\infty} e^{\iota n}, \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{e^{\iota n\theta}}{n}.$$

Problem 18. Let $\sum_{n=0}^{\infty} a_n z^{n+1}$ be a power series with non–zero radius of convergence. Show that $\displaystyle\sum_{n=0}^{\infty} a_n \frac{t^n}{n!}$ has infinite radius of convergence.

Problem 19. *Bernoulli numbers.* Define $\{B_n\}_{n=0}^{\infty} \subset \mathbb{Q}$ by:

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \text{ for } |z| < 2\pi.$$

(a) Show that $B_0 = 1$, $B_1 = -\frac{1}{2}$ and $B_{2k+1} = 0$, for all $k \geq 1$.
(b) Prove the following recurrence relation for $\{B_n\}$:

$$B_n = -\frac{1}{n+1} \sum_{\ell=0}^{n-1} \binom{n+1}{\ell} B_\ell \; \forall \; n \geq 1.$$

(c) Write the Taylor series expansion of $\log(\sin(z)/z)$, near $z = 0$, and indicate its radius of convergence.

Problem 20. Let $A(z) = \displaystyle\sum_{k=0}^{\infty} a_k z^k$. Prove that $A(z)$ is the Taylor series expansion, near 0, of a rational function (defined at 0) if, and only if there exist $q_1, \ldots, q_\ell \in \mathbb{C}$ and $N \geq \ell$ such that

$$a_n = q_1 a_{n-1} + q_2 a_{n-2} + \cdots + q_\ell a_{n-\ell}, \text{ for every } n \geq N.$$

Problem 21. Let $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n z^n$ be the Taylor series of a holomorphic function, for all $z \in D(0; R)$.

(a) Show that for every $r < R$, we have:

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{\iota\theta})|^2 \, d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n} .$$

(b) For each $r < R$, let $M(r) := \text{Max}\{|f(s)| : |s| = r\}$. Prove that $|c_n| \leq \dfrac{M(r)}{r^n}$, for every $n \geq 0$.
(c) Show that, if there exists $r \in (0, R)$ and $n \in \mathbb{Z}_{\geq 0}$ such that $|c_n| r^n = M(r)$, then $f(z) = c_n z^n$.

Problem 22. For a holomorphic function $f : \Omega \to \mathbb{C}$ and an isolated singularity α of f (i.e., $D^\times(\alpha, r) \subset \Omega$ for some $r > 0$), define:

$$\mathrm{Res}(f;\alpha) := \frac{1}{2\pi\iota}\int_{C(\alpha,r)} f(z)\, dz\,.$$

(a) If $f(z) = \sum_{m=1}^{\infty} d_m(z-\alpha)^{-m} + \sum_{n=0}^{\infty} c_n(z-\alpha)^n$ is the Laurent series expansion of f on the punctured neighbourhood $D^\times(\alpha, r)$, then show that $\mathrm{Res}(f;\alpha) = d_1$.
(b) Let $\varphi : D(\alpha, r) \to \mathbb{C}$ be another, holomorphic function. Show that $\mathrm{Res}(\varphi f;\alpha) = \varphi(\alpha)\mathrm{Res}(f;\alpha)$.
(c) Same φ as above, assume that α is a pole of order N, for the function f. Show that:

$$\mathrm{Res}\left(\varphi\frac{f'}{f};\alpha\right) = N\varphi(\alpha)$$

Problem 23. For each of the following, determine a formal solution $w(z)$, near $z = 0$, with $w(0) = 0$ and $w'(0) = 1$ initial conditions. Discuss the radius of convergence of your solution.
(a) $w'' - z^2w = 3z^2 - z^4$.
(b) $(1 - z^2)w'' - 2zw' + n(n + 1)w = 0$, here $n \in \mathbb{Z}$.

Problem 24. Let X be an $n \times n$ matrix over $\mathbb{C}$, and let $\sigma(X) = \{\lambda_j\}_{j=1}^n \subset \mathbb{C}$ be the eigenvalues of X (repeated according to their multiplicities).
(1) Consider the operator $\mathrm{ad}(X) : Y \mapsto XY - YX$ on the vector space of $n \times n$ matrices. Show that the eigenvalues of $\mathrm{ad}(X)$ are $\{\lambda_j - \lambda_k : 1 \le j, k \le n\}$.

(2) Let $f : \Omega \to \mathbb{C}$ be a holomorphic function, where $\Omega \subset \mathbb{C}$ is an open, connected set containing $\sigma(X)$. Define

$$f(X) := \frac{1}{2\pi\iota}\int_C (z - X)^{-1} f(z)\, dz\,,$$

where C is a contour in Ω containing $\sigma(X)$. Show that the eigenvalues of $f(X)$ are $\{f(\lambda_j) : 1 \le j \le n\}$.

(3) Let $\|\cdot\|$ denote a norm on the space of $n \times n$ matrices. Prove the following (Gelfand's formula)

$$\lim_{n\to\infty} \|X^n\|^{\frac{1}{n}} = \mathrm{Max}\{|\lambda_j| : 1 \le j \le n\}.$$

Problem 26. Let $f : \mathbb{R}_{>0} \to \mathbb{C}$ be a continuous function such that (i) $f(t) \to 0$ as $t \to 0^+$, and (ii) there exist constants $M, C, R \in \mathbb{R}_{>0}$ such that $|f(t)| < Me^{Ct}$ for all $t > R$. Show that, for $z \in \mathbb{C}$ such that $\mathrm{Re}(z) > 0$, we have:

$$\int_0^\infty f(t)e^{-zt}\, dt = z^{-1}\int_0^\infty f'(t)e^{-zt}\, dt.$$

Use this to prove the identity (Euler): $\displaystyle\int_0^\infty \frac{t^n}{n!}e^{-zt}\, dt = z^{-n-1}$, for all $n \in \mathbb{Z}_{\ge 0}$.

Problem 27. Compute the asymptotic expansions of the following integrals, as $\mathrm{Re}(z) \to \infty$.

$$\text{(a)} \int_0^\infty \frac{e^{-zt}}{1+t^n}\,dt \ (n \in \mathbb{Z}_{\geq 1}), \qquad \text{(b)} \int_0^\infty (1+t)^a e^{-zt}\,dz \ (a \in \mathbb{C}).$$

Problem 28. Let $g : \mathbb{C} \dashrightarrow \mathbb{C}$ be a meromorphic function, with finitely many poles at $a_1, \ldots, a_n \in \mathbb{C}$. Let $A \in \mathbb{R}$ be such that $\mathrm{Re}(a_j) < A$ for all $1 \leq j \leq n$. Assume that $\lim_{\substack{z\to\infty \\ \mathrm{Re}(z)\leq A}} |g(z)| = 0$. Prove that, for every $t \in \mathbb{R}_{>0}$, we have:

$$\int_{L_A} g(z)e^{zt}\,dz = 2\pi\iota \sum_{k=1}^{n} \mathrm{Res}_{z=a_j}(f(z)e^{zt}).$$

Here, $L_A = A + \iota\mathbb{R}$ is the infinite vertical line.

Problem 29. Let $a \in \mathbb{R}_{>0}$ and $L_a = a + \iota\mathbb{R}$. Show that for every $n \in \mathbb{Z}_{\geq 0}$ and $t \in \mathbb{R}_{>0}$, we have:

$$\frac{1}{2\pi\iota}\int_{L_a} e^{zt}z^{-n-1}\,dz = \frac{t^n}{n!}\,.$$

(Bonus) Generalize this to obtain the inverse to the Laplace transform.

Problem 30. Compute the Laplace transform $\displaystyle\int_0^\infty \varphi(t)e^{-zt}\,dt$ for $\varphi : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ given by $\varphi(t) = n$ for $t \in [n, n+1)$.

Problem 31. Let $\psi(x + \iota y) = u(x, y) + \iota v(x, y)$ be a holomorphic function on an open, connected set $\Omega \subset \mathbb{C}$. Let $z_0 = x_0 + \iota y_0 \in \Omega$ be a critical point of ψ (i.e., $\psi'(z_0) = 0$) and let $n \in \mathbb{Z}_{\geq 2}$ be such that $0 = \psi'(z_0) = \ldots = \psi^{(n-1)}(z_0)$ and $\psi^{(n)}(z_0) = ae^{\iota\alpha} \neq 0$.

(1) Write $z = x + \iota y = z_0 + \rho e^{\iota\theta}$, $u = u(x, y)$, $v = v(x, y)$, $u_0 = u(x_0, y_0)$ etc. Show that

$$u^2 + v^2 = u_0^2 + v_0^2 + \frac{2a\rho^n}{n!}\left(u_0\cos(n\theta + \alpha) + v_0\sin(n\theta + \alpha)\right) + O(\rho^{n+1}), \quad \text{as } \rho \to 0^+\,.$$

(2) Prove that, through z_0, the directions of steepest ascent/descent are given as follows.

Steepest descent	$-\frac{\alpha}{n} + (2p+1)\frac{\pi}{n}$	$p = 0, \ldots, n-1$
Steepest ascent	$-\frac{\alpha}{n} + 2p\frac{\pi}{n}$	$p = 0, \ldots, n-1$

(3) Prove the maximum modulus principle: if $|\psi(z)|$ takes its maximum value at $z_0 \in \Omega$, then ψ is a constant function.

Problem 32. Compute the critical points and directions of steepest ascent/descent from each of these points, for the following functions: (a) $\dfrac{z^2}{2} - \alpha z$, $(\alpha \in \mathbb{C})$, and (b) $\dfrac{z^3}{3} - z$.

Problem 33. Prove the Riemann-Lebesgue Lemma: if $(a, b) \subset \mathbb{R}$ and $f \in L^1(a, b)$ (that is, $\displaystyle\int_a^b |f(t)|\,dt < \infty$), then

$$\lim_{\lambda\to+\infty}\int_a^b f(t)e^{\iota\lambda t}\,dt = 0\,.$$

Problem 34. Prove the following asymptotic expansion, as $x \to \infty$:

$$\int_0^1 \ln(t) e^{\iota x t}\, dt \sim \iota \frac{\ln(x)}{x} - \frac{\iota\gamma + \pi/2}{x} + \iota e^{\iota x} \sum_{n=1}^{\infty} (-1)^n \frac{(n-1)!}{x^{n+1}} \ .$$

Here, γ is the Euler–Mascheroni constant.

Problem 35. Discuss the $n \to \infty$ behaviour of $\Psi^{(n)}(1)$, where $\Psi(z) = \dfrac{\Gamma'(z)}{\Gamma(z)}$ is the logarithmic derivative of the gamma function.

Problem 36. Assume that $\{a_n\}_{n=0}^{\infty} \subset \mathbb{C}$ is a sequence of complex numbers such that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire (i.e., radius of convergence is infinity) function. Note that, by Cauchy's formula:

$$a_n = \frac{1}{2\pi\iota} \int_C z^{-n-1} f(z)\, dz = \frac{1}{2\pi\iota} \int_C e^{-\ln(z)(n+1)} f(z)\, dz \ .$$

Use this observation, together with the method of steepest descent, to obtain $n \to \infty$ behaviour of a_n, in the following examples.

(a) $f(z) = e^z$, (b) $f(z) = e^{z^k}$, (c) $a_n = \frac{B_n}{n!}$, where B_n is the n-th Bernoulli number (see Problem 19 above).

Problem 37. Discuss the change in asymptotic expansions as $\lambda \to \infty$ of the following integral, as θ varies in $(0, 2\pi)$, $\theta \neq \pi$.

$$I(\lambda, \theta) = \int_0^{e^{\iota\theta}} \frac{e^{\lambda z^2}}{1+z}\, dz \, .$$

Problem 38. Compute the leading term behaviour of the following integral, as $\lambda \to \infty$. Here $a \in \mathbb{R}$ is a fixed constant, and C is the infinite horizontal line $\mathbb{R} + \iota$.

$$I(\lambda, a) = \int_C \frac{e^{\iota\lambda\left(\frac{z^3}{3} - z\right)}}{z - a}\, dz \, .$$