Lecture 1.

Lie algebras – definitions

1. **Definition** - A *Lie algebra* $\mathfrak{g}$ over a field k is a k-vectorspace together with a bilinear map (called *Lie bracket*)

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$
$$(a,b) \mapsto [a,b]$$

satisfying the following two axioms:

(Skew-symmetry) $\quad [x,x] = 0 \quad \forall\ x \in \mathfrak{g}.$

(Jacobi identity) $\quad [x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$
$\qquad\qquad\qquad\qquad\qquad\qquad \forall\ x,y,z \in \mathfrak{g}.$

Remark 1:- when $\mathrm{char}(k) \neq 2$, (Skew-symmetry) axiom is equivalent to $[x,y] = -[y,x]\ \ \forall\ x,y \in \mathfrak{g}.$

(Pf.- if $[x,x] = 0\ \forall\ x \in \mathfrak{g}$, then $0 = [x+y, x+y]$
$= [x,x] + [y,y] + [x,y] + [y,x]$

$\Rightarrow \quad [x,y] = -[y,x]$ for every $x,y \in \mathfrak{g}$.

Conversely, if $[x,y] = -[y,x]$ for every $x,y \in \mathfrak{g}$, then take $x=y$ to get $2[x,x] = 0$ implying that $[x,x] = 0$ when $\mathrm{char}(k) \neq 2$. $\square$)

Remark 2. - Lie algebras arise naturally from Lie groups - which are groups whose underlying set has a structure of a smooth manifold - (not to be discussed much in this course, but the one on Lie groups next semester) so that multiplication and inverse are smooth maps.

e.g. S^1, $GL_n(\mathbb{C})$, $SO_n(\mathbb{R})$ etc.

Sophus Lie (1842-1899) coined the term "infinitesimal symmetries/transformations" - later replaced by Lie algebras in the works (~1870) of Herman Weyl.

Lie's theorem. - G : Lie group $\rightsquigarrow$ $\mathfrak{g} = T_eG$ (tangent space at neutral elt. $e \in G$) has a Lie alg. structure

subgroups / normal subgroups ... $\rightsquigarrow$ subalg's / ideals ...

Lie group hom's $\rightsquigarrow$ Lie alg. hom's. etc.

2. Examples. - (a) Abelian Lie algebra : Lie bracket is identically zero

$$[x,y] = 0 \quad \forall x, y \in \mathfrak{g}.$$

(b) $\mathfrak{gl}_n(k)$ = vector space of all $n \times n$ matrices over k (field)

$[X,Y] := XY - YX$ matrix commutator.

(Pf. of Jacobi identity: $[x,[y,z]] = x(yz-zy) - (yz-zy)x$

$= xyz - xzy - yzx + zyx$

So, left-hand side of Jacobi's identity becomes - symmetrize over cyclic reordering of x, y, z.

$$\begin{aligned} &xyz - xzy - yzx + zyx \\ &+ yzx - yxz - zxy + xzy \\ &+ zxy - zyx - xyz + yxz \end{aligned} \quad = 0 \quad .$$

(c) A bit more generally, let A be an associative algebra over k. Then $\mathrm{Lie}(A) :=$ A as vector space, with Lie bracket given by commutator $[a,b] := ab - ba$.

Thus $\mathfrak{gl}_n(k) = \mathrm{Lie}\left(\underbrace{M_{n\times n}(k)}_{\text{unital, assoc. alg. of } n\times n \text{ matrices over } k} \right)$

or, replacing k^n by a vector space V over k -

$$\mathfrak{gl}(V) := \mathrm{Lie}\left(\mathrm{End}_k(V) \right)$$

(d) Derivations of an algebra.- Let B be an arbitrary algebra (not necessarily unital or associative) over k.

$$\mathrm{Der}(B) := \{ f : B \to B \ k\text{-linear map satisfying } f(xy) = f(x)y + x f(y) \text{ - Leibniz id.} \quad \text{for every } x, y \in B \}$$

$$[\cdot,\cdot] : \mathrm{Der}(B) \times \mathrm{Der}(B) \longrightarrow \mathrm{Der}(B)$$
$$[d_1, d_2] = d_1 d_2 - d_2 d_1$$

Claim.- if $d_1, d_2 \in \mathrm{Der}(B)$, then $[d_1, d_2] \in \mathrm{Der}(B)$.
(careful: $d_1 d_2$, $d_2 d_1$ may not be in $\mathrm{Der}(B)$)

Pf.- $d_1(d_2(xy)) = d_1\left(d_2(x)\, y + x\, d_2(y) \right)$
$$= d_1(d_2(x))\, y + d_2(x)\, d_1(y) + d_1(x)\, d_2(y) + x\, d_1(d_2(y))$$

$\Rightarrow [d_1, d_2](xy) = [d_1, d_2](x) \cdot y + x \cdot [d_1, d_2](y)$. $\square$

3. Homomorphisms. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two Lie algebras over k.

A linear map $f: \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie alg. homomorphism if

$$f([x,y]_1) = [f(x), f(y)]_2 \quad \forall\, x, y \in \mathfrak{g}_1.$$

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{r} \subset \mathfrak{g}$ be a vector subspace. We say $\mathfrak{r}$ is a Lie subalgebra (resp. ideal) if

$$[x,y] \in \mathfrak{r} \quad \text{for every} \quad x, y \in \mathfrak{r}$$

$$(\text{resp. for every } x \in \mathfrak{g},\ y \in \mathfrak{r}).$$

Theorem.- If $f: \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie alg. homomorphism, then

$$\operatorname{Ker}(f) := \{x \in \mathfrak{g}_1 : f(x) = 0\} \subset \mathfrak{g}_1 \text{ is an ideal}$$

$$\operatorname{Im}(f) := \{f(x) : x \in \mathfrak{g}_1\} \subset \mathfrak{g}_2 \text{ is a Lie subalgebra}$$

and f induces an isomorphism

$$\mathfrak{g}_1 / \operatorname{Ker}(f) \xrightarrow{\sim} \operatorname{Im}(f)$$

Remark.- By skew-symmetry of the Lie bracket, every left ideal is automatically a right ideal, so we just say "ideal".

If $\mathfrak{r} \subset \mathfrak{g}$ is an ideal, the quotient space $\mathfrak{g}/\mathfrak{r}$ inherits a Lie algebra structure from $\mathfrak{g}$

$$[x \pmod{\mathfrak{r}},\ y \pmod{\mathfrak{r}}] := [x,y] \pmod{\mathfrak{r}}.$$

The details of this, and the proof of the isomorphism theorem above, are left as a routine exercise.

4. $\mathfrak{g}$-representations. - Let $\mathfrak{g}$ be a Lie algebra over k. A representation of $\mathfrak{g}$ is a k-vector space V together with a linear map

$$\rho : \mathfrak{g} \to \mathrm{End}(V) \quad \text{so that} \quad \rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$$
$$\forall\ x, y \in \mathfrak{g}.$$

In other words, ρ is a Lie alg. hom $\mathfrak{g} \to \mathfrak{gl}(V)$. Often we suppress ρ from the notations and simply write "V is a $\mathfrak{g}$-repn." or $\mathfrak{g} \curvearrowright V$, and for $X \in \mathfrak{g}$, $v \in V$

$$X \cdot v = \rho(X)(v).$$

5. Examples. - (a) The following are Lie subalgebras of $\mathfrak{gl}_n(k)$ (but not ideals)

$\mathfrak{h}_n(k)$ = subspace of diagonal matrices

$$= \left\{ \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} : \lambda_1, \ldots, \lambda_n \in k \right\}$$

$\mathfrak{b}_n(k)$ = subspace of upper triangular matrices

$$= \left\{ \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ 0 & a_{22} \cdots & a_{2n} \\ & \vdots & \vdots \\ 0 & \cdots & 0\ a_{nn} \end{bmatrix} : \begin{array}{l} a_{ij} \in k \\ 1 \le i \le j \le n \end{array} \right\}$$

$\mathfrak{u}_n(k)$ = subspace of strict upper triangular matrices

$$\mathfrak{u}_n(k) = \left\{ \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ & 0 & a_{23} & \cdots & a_{2n} \\ & & \ddots & & \vdots \\ & & & & a_{n-1,n} \\ & & & & 0 \end{bmatrix} : \begin{array}{l} a_{ij} \in k \\ 1 \leq i < j \leq n \end{array} \right\}$$

Note: $\mathfrak{u}_n(k) \subset \mathfrak{b}_n(k)$ is an ideal

(b) Consider Example (d) of §2 above: B is an arbitrary alg./k

$\mathrm{Der}(B) \subset \mathfrak{gl}(B)$ is a subalgebra (Lie).

(c) $\mathfrak{Z}(\mathfrak{g}) := \{ x \in \mathfrak{g} : [x,y] = 0 \ \forall\, y \in \mathfrak{g} \}$ called the center of the Lie algebra $\mathfrak{g}$, is an ideal in $\mathfrak{g}$.

(d) <u>Commutator ideal</u> - Let $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$ be two ideals

$$[\mathfrak{a}, \mathfrak{b}] := k\text{-span of } \{ [a,b] : \begin{array}{l} a \in \mathfrak{a} \\ b \in \mathfrak{b} \end{array} \} \subset \mathfrak{g}$$

is an ideal. When $\mathfrak{a} = \mathfrak{b} = \mathfrak{g}$, the resulting ideal $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$ is called <u>the commutator ideal</u>.

Ex.- Show that $[\mathfrak{g}, \mathfrak{g}]$ is the smallest ideal I so that $\mathfrak{g}/I$ is abelian.

Ex. - If $\mathfrak{g}_1 \xrightarrow{f} \mathfrak{g}_2$ is a Lie alg. hom. and $\mathfrak{g}_2$ is abelian, then $[\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathrm{Ker}(f)$.

6. Jacobi identity and adjoint representation - Let $\mathfrak{g}$ be a Lie algebra. Define $ad : \mathfrak{g} \to End_k(\mathfrak{g})$ by

$$ad(X) : Y \mapsto [X, Y] \quad , \quad \forall\ X, Y \in \mathfrak{g}.$$

Lemma.- (a) $ad(X) \in Der_k(\mathfrak{g}) \quad \forall\ X \in \mathfrak{g}.$

(b) $ad : \mathfrak{g} \to Der_k(\mathfrak{g}) \subset \mathfrak{gl}_k(\mathfrak{g})$ is a Lie alg. hom.

That is, ad is a representation of $\mathfrak{g}$ - called the adjoint representation.

Proof.- (a) $ad(X) : \mathfrak{g} \to \mathfrak{g}$ is a derivation if and only if
$Y \mapsto [X, Y]$

$$ad(X) \cdot ([Y, Z]) = [ad(X) \cdot (Y), Z] + [Y, ad(X) \cdot (Z)]$$

for every $X, Y, Z \in \mathfrak{g}$. This equation is same as

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$$

which becomes Jacobi identity, using skew-symmetry.

(b) To show : $ad([X, Y]) = ad(X)\, ad(Y) - ad(Y)\, ad(X)$
for every $X, Y \in \mathfrak{g}$.

Evaluate on a typical $Z \in \mathfrak{g}$, to get

$$[[X, Y], Z] = [X, [Y, Z]] - [Y, [X, Z]]$$

which is again Jacobi identity - given skew-symmetry. □