Lecture 2

1. Recall - in the previous lecture, we defined the following objects:

Lie algebras, subalgebras, ideals, homomorphisms of Lie algebras, representations of a Lie algebra

Direct sum and semidirect product - Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two Lie algebras. The following Lie bracket on the vector space $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ turns it into a Lie algebra - called the direct sum of $\mathfrak{g}_1$ and $\mathfrak{g}_2$:

$$[(a_1, a_2), (b_1, b_2)] = ([a_1, b_1]_{\mathfrak{g}_1}, [a_2, b_2]_{\mathfrak{g}_2})$$

$\forall\ a_1, b_1 \in \mathfrak{g}_1$ and $a_2, b_2 \in \mathfrak{g}_2$.

Lemma. - Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{g}_1 \subset \mathfrak{g}$ and $\mathfrak{g}_2 \subset \mathfrak{g}$ be two subalgebras. Then $\mathfrak{g} \simeq \mathfrak{g}_1 \oplus \mathfrak{g}_2$ as Lie algebras iff

(1) $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ as vector spaces. (i.e. $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ & $\mathfrak{g}_1 \cap \mathfrak{g}_2 = \{0\}$)

(2) $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are ideals in $\mathfrak{g}$.

Pf. - The forward implication is clear. For the converse, let $\mathfrak{h} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be the direct sum of Lie algebras and let $f : \mathfrak{h} \to \mathfrak{g}$ be the vector space isomorphism. We will show that f is a homomorphism of Lie algebras:

$$f([x, y]_{\mathfrak{h}}) = [f(x), f(y)]_{\mathfrak{g}} \qquad \forall\ x, y \in \mathfrak{h} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

This is true by definition if $x, y \in \mathfrak{g}_1$ or $x, y \in \mathfrak{g}_2$.

If $x \in \mathfrak{g}_1$ and $y \in \mathfrak{g}_2$ then $[x, y]_{\mathfrak{h}} = 0$. As $\mathfrak{g}_1 \hookrightarrow \mathfrak{g}$ and $\mathfrak{g}_2 \hookrightarrow \mathfrak{g}$ are ideals, $[x, y]_{\mathfrak{g}} \in \mathfrak{g}_1 \cap \mathfrak{g}_2 = \{0\}$ and the equation holds. □

e.g. $\mathfrak{gl}_n(k) = k \cdot I_n \oplus \mathfrak{sl}_n(k)$ [if $\mathrm{char}(k)$ does not divide n]

where $\mathfrak{sl}_n(k) = \{ X \in \mathfrak{gl}_n(k) : \mathrm{Tr}(X) = 0 \}$

is a direct sum of Lie algebras.

2. Semidirect product .- Again, let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two Lie algebras.

Assume given a Lie alg. hom. $\mathfrak{g}_1 \xrightarrow{\alpha} \mathfrak{gl}(\mathfrak{g}_2)$

so that $\alpha(X) : \mathfrak{g}_2 \to \mathfrak{g}_2$ is a derivation - i.e.

$$\alpha(X)([Y,Z]) = [\alpha(X)(Y), Z] + [Y, \alpha(X)(Z)] \qquad \forall\ X \in \mathfrak{g}_1,\ Y, Z \in \mathfrak{g}_2.$$

In other words, $\alpha : \mathfrak{g}_1 \longrightarrow \mathrm{Der}(\mathfrak{g}_2) \subset \mathfrak{gl}(\mathfrak{g}_2)$

Lie algebra hom.

Defn.- $\mathfrak{g} = \mathfrak{g}_1 \ltimes \mathfrak{g}_2$ is defined as $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ as a vector space with Lie bracket given by:

$$[x,y]_{\mathfrak{g}} = \begin{cases} [x,y]_{\mathfrak{g}_1} & \text{if } x, y \in \mathfrak{g}_1 \\ [x,y]_{\mathfrak{g}_2} & \text{if } x, y \in \mathfrak{g}_2 \\ \alpha(x)(y) & \text{if } x \in \mathfrak{g}_1,\ y \in \mathfrak{g}_2 \end{cases}$$

i.e. $[x_1 + x_2, y_1 + y_2]_{\mathfrak{g}} = [x_1, y_1]_{\mathfrak{g}_1} + [x_2, y_2]_{\mathfrak{g}_2} + \alpha(x_1)(y_2) - \alpha(y_1)(x_2)$

$x_1, y_1 \in \mathfrak{g}_1$

$x_2, y_2 \in \mathfrak{g}_2$

Note: $\mathfrak{g}_2$ is an ideal in $\mathfrak{g}$ and $\mathfrak{g}_1$ is a subalgebra.

We have the following analogue of the lemma of §1 above.

Lemma. - Let $\mathfrak{g}$ be a Lie algebra, or, $\mathfrak{b} \subset \mathfrak{g}$ be subalgebras.

Then $\mathfrak{g} \simeq \mathfrak{or} \ltimes \mathfrak{b}$ if and only if

(i) $\mathfrak{g} = \mathfrak{or} \oplus \mathfrak{b}$ as vector spaces

(ii) $\mathfrak{b} \subset \mathfrak{g}$ is an ideal

$\mathfrak{or} \xrightarrow{\alpha} \mathrm{Der}(\mathfrak{b}) \subset \mathfrak{gl}(\mathfrak{b})$ is given by $\alpha(X)(Y) = [X, Y]$ $X \in \mathfrak{or}$, $Y \in \mathfrak{b}$.

Proof is left as an easy exercise.

e.g. $$\mathfrak{b}_n(k) = \left\{ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix} : a_{ij} \in k \quad 1 \leq i \leq j \leq n \right\}$$

$$\simeq \mathfrak{h}_n(k) \ltimes \mathfrak{u}_n(k)$$ see Example (a) of §5 Lecture 1.

3. Operations with representations

- Let $\mathfrak{g}$ be a Lie algebra; and let V, V_1, V_2 be representations of $\mathfrak{g}$

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V), \quad \rho_\ell : \mathfrak{g} \to \mathfrak{gl}(V_\ell) \quad \ell = 1, 2$$

(i) $V_1 \oplus V_2$ is again a $\mathfrak{g}$-representation via

$$\mathfrak{g} \longrightarrow \mathfrak{gl}(V_1 \oplus V_2)$$

$$X \longmapsto \begin{bmatrix} \rho_1(X) & 0 \\ 0 & \rho_2(X) \end{bmatrix} \text{ or } \rho_1(X) \oplus \rho_2(X)$$

called the direct sum of representations V_1 & V_2.

(ii) $\underset{k\text{-v.s.}}{\mathrm{Hom}}(V_1, V_2)$ is again a $\mathfrak{g}$-representation via

$$\alpha : \mathfrak{g} \longrightarrow \mathfrak{gl}\left(\mathrm{Hom}_k(V_1, V_2)\right)$$

$$\alpha(x)\cdot f \;:\; v_1 \longmapsto \rho_2(x)\left(f(v_1)\right) - f\left(\rho_1(x)(v_1)\right)$$

$\forall\, x \in \mathfrak{g}$, $f : V_1 \to V_2$ linear, $v_1 \in V_1$.

Remark. - A linear map $f : V_1 \to V_2$ is a <u>homomorphism</u> of <u>$\mathfrak{g}$-representations</u>* if $f\left(\rho_1(x)(v)\right) = \rho_2(x)\left(f(v)\right)$ $\forall\, x \in \mathfrak{g}$, $v \in V_1$

i.e. $\alpha(x)\cdot f = 0 \quad \forall\, x \in \mathfrak{g}$

$\mathrm{Hom}_{\mathfrak{g}}(V_1, V_2) :=$ vector space of hom's of $\mathfrak{g}$-representations $\subset \mathrm{Hom}_k(V_1, V_2)$

(iii) Special case of (ii) when $V_2 = \mathbb{C}$ k (base field) is the <u>trivial</u> one-dimensional representation of $\mathfrak{g}$

i.e. $\rho_2(x) = 0 \;\; \forall\, x \in \mathfrak{g}$; $V_1 = V$, so that

$\mathrm{Hom}_k(V, k) = V^*$ dual vector space.

$\mathfrak{g}$ acts on V^* via:

$$(X \cdot \xi)(v) = -\,\xi(X \cdot v)$$

$\forall\, \xi \in V^*$, $X \in \mathfrak{g}$, $v \in V$

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* also, sometimes called $\mathfrak{g}$-linear map or $\mathfrak{g}$-intertwiner

(iv) Tensor product of two representations :

$\mathfrak{g}$ acts on $V_1 \otimes V_2$ via

$$X \cdot (v_1 \otimes v_2) = (X \cdot v_1) \otimes v_2 + v_1 \otimes (X \cdot v_2)$$

for every $X \in \mathfrak{g}$, $v_1 \in V_1$ & $v_2 \in V_2$.

Ex.- Let V, W be two $\mathfrak{g}$-representations. Show that the natural linear map $V^* \otimes W \xrightarrow{A} \mathrm{Hom}_{\mathbb{K}}(V, W)$

$$A(\xi \otimes w) : v \longmapsto \xi(v)\, w \quad ; \quad \begin{pmatrix} \xi \in V^* \\ w \in W \\ v \in V \end{pmatrix}$$

is a homomorphism of $\mathfrak{g}$-representations.

Ex. Let $n \in \mathbb{Z}_{\geq 2}$ and V be a $\mathfrak{g}$-representation. Let $\sigma \in S_n$ (symmetric group) act on $V^{\otimes n} = \underbrace{V \otimes V \otimes \dots \otimes V}_{n\text{-fold}}$

via $\sigma \cdot (v_1 \otimes \dots \otimes v_n) = v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}$

Then $\sigma : V^{\otimes n} \to V^{\otimes n}$ is a $\mathfrak{g}$-intertwiner.

Ex. Show that $V \otimes V \simeq \mathrm{Sym}^2(V) \oplus \Lambda^2(V)$ as $\mathfrak{g}$-representations.

4. Indecomposable and irreducible representations -

Let $\mathfrak{g}$ be a Lie algebra and V a representation. We say

that V is <u>irreducible</u>* if the only <u>proper subrepresentation</u> of V is $\{0\}$. That is, for a subspace $V' \subset V$,

$$\left[\begin{array}{l} \text{if } x \cdot v \in V' \text{ for every } x \in \mathfrak{g} \text{ and } v \in V', \text{ then} \\ \quad \text{either } V' = V \text{ or } V' = \{0\} \end{array}\right]$$

We say V is <u>indecomposable</u> if V is not isomorphic to a direct sum of non-zero representations. That is,

$$\left[\begin{array}{l} \text{if } V_1, V_2 \subset V \text{ are two subrepresentations s.t.} \\ \quad V_1 \oplus V_2 \simeq V, \text{ then either } V_1 = \{0\} \\ \qquad\qquad\qquad\qquad \text{or } V_2 = \{0\} \end{array}\right]$$

Remark. - Every irreducible representation is indecomposable, but the converse is not true in general.

e.g. $\mathfrak{g} = b_2(k) = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a, b, d \in k \right\} \hookrightarrow \mathfrak{gl}_2(k)$

$\mathfrak{g}$ is indecomposable but not irreducible.

$\mathfrak{gl}_2(k)$: 2-dim'l repn. of $b_2(k)$

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* Zero vector space is not considered irreducible.