Lecture 3

1. Recall - last time we defined the following operations with representations of a Lie algebra $\mathfrak{g}$ (over a field k):

$$V_1, V_2 : \mathfrak{g}\text{-repns} \rightsquigarrow V_1 \oplus V_2,\ V_1 \otimes V_2,\ \operatorname{Hom}_{k\text{-v.s.}}(V_1, V_2)$$

have a natural structure of $\mathfrak{g}$-repns.

We also introduced the notions of indecomposable and irreducible representations.

Schur's lemma. - Let $\mathfrak{g}$ be a Lie algebra and $f: V \to W$ be a homomorphism of $\mathfrak{g}$-representations.

(1) If V is irreducible then either $f \equiv 0$ or f is injective.

(2) If W is irreducible then either $f \equiv 0$ or f is surjective.

(3) Assume that k is algebraically closed and V, W are finite-dimensional and irreducible. Then

$$\operatorname{Hom}_{\mathfrak{g}}(V, W) = \begin{cases} \{0\} & \text{if } V \text{ and } W \text{ are isomorphic} \\ 1\text{-dim'l} & \text{if } V \simeq W \\ (= k \cdot \mathrm{Id}_V) & \end{cases}$$

Proof. - (1) Since $\operatorname{Ker}(f) \subset V$ is a subrepresentation, and V is irreducible, either $\operatorname{Ker}(f) = \{0\}$ or $\operatorname{Ker}(f) = V$, ie. either f is injective or $f \equiv 0$.

(2) Same argument as above for $\operatorname{Im}(f) \subset W$.

(3) If V and W are irreducible then either $\mathrm{Hom}_{\mathfrak{g}}(V,W)=\{0\}$, or any non-zero $\mathfrak{g}$-intertwiner is an isomorphism. Thus if $\mathrm{Hom}_{\mathfrak{g}}(V,W)\neq\{0\}$, we can fix an isomorphism and assume $V=W$. If $\dim(V)<\infty$, $k=\bar{k}$, and $f\in\mathrm{Hom}_{\mathfrak{g}}(V,V)$ is non-zero, then let $\lambda\in k$ ($\lambda\neq 0$) be an eigenvalue of f. Thus, $\mathrm{Ker}(f-\lambda\cdot \mathrm{Id}_V)\subset V$ is a non-zero representation hence $V=\mathrm{Ker}(f-\lambda\,\mathrm{Id}_V)$ by irreducibility of V. So $f=\lambda\cdot\mathrm{Id}_V$.

2. Examples. - (a) Let $\mathfrak{g}=\mathfrak{gl}_n(k)$, $V=k^n$ is naturally a $\mathfrak{g}$-representation. Verify that V is irreducible.

For any $\ell\in\mathbb{Z}_{\geq 0}$, show that $\Lambda^{\ell}(V)$ is irreducible $\forall\ \ell\leq n$.

Remark. - for $\ell=n$, $\Lambda^n(V)\simeq k$ is 1-dimensional with $\mathfrak{g}$-action given by $\mathfrak{gl}_n(k)\longrightarrow \mathfrak{gl}(\Lambda^n(V))=\mathfrak{gl}_1(k)=k$

$$X\longmapsto \mathrm{Trace}(X).$$

(b) If $\mathfrak{a}$ is an abelian Lie algebra, k is algebraically closed, and V is a finite-dimensional irreducible representation of $\mathfrak{a}$, then $\dim V=1$

Pf. - Replacing $\mathfrak{a}$ by the image of the action homomorphism $\rho:\mathfrak{a}\to\mathfrak{gl}(V)$, we may assume that $\mathfrak{a}$ is finite-dim'l.

In other words there are finitely many $x_1, \dots, x_n \in \mathcal{A}$ s.t. $\rho(\mathcal{A}) = \text{Span of } \{\rho(x_1), \dots, \rho(x_n)\}$. As $\mathcal{A}$ is abelian, these operators commute. As k is alg. closed & $\dim(V) < \infty$, we can find a joint eigenvector $0 \neq v \in V$, i.e. $\rho(x_j).v = \lambda_j v \quad \forall\ 1 \leq j \leq n$

$\lambda_1, \dots, \lambda_n \in k$.

Thus $kv \subset V$ is a non-zero subrepresentation, hence $V = kv$ by irreducibility of V. □

(c) For a Lie algebra $\mathfrak{g}$, the set of 1-dim'l representations is in bijection with $\left(\mathfrak{g}/_{[\mathfrak{g},\mathfrak{g}]}\right)^*$ where $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{g}$ is the commutator ideal, and $(\ )^*$ means linear dual $\underset{k\text{-v.s}}{\text{Hom}}(-, k)$.

Pf.- if $\rho : \mathfrak{g} \to \mathfrak{gl}_1(k) = k$ is a Lie alg. hom then

$$\rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x) = 0 \quad \forall\ x, y \in \mathfrak{g}.$$

Hence $[\mathfrak{g},\mathfrak{g}] \subset \text{Ker}(\rho)$ and ρ defines $\mathfrak{g}/_{[\mathfrak{g},\mathfrak{g}]} \xrightarrow{\bar{\rho}} k$ linear map. Conversely, since $\mathfrak{g}/_{[\mathfrak{g},\mathfrak{g}]}$ is abelian,

$$\underset{\text{Lie Alg}}{\text{Hom}}\left(\mathfrak{g}/_{[\mathfrak{g},\mathfrak{g}]}, \mathfrak{gl}_1(k)\right) = \underset{k\text{-v.s.}}{\text{Hom}}\left(\mathfrak{g}/_{[\mathfrak{g},\mathfrak{g}]}, k\right)$$

and we are done. □

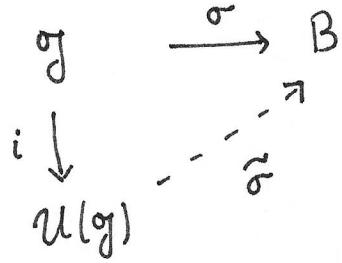
3. Universal enveloping algebra of a Lie algebra - Let $\mathfrak{g}$ be a Lie algebra over a field k. The universal enveloping algebra of $\mathfrak{g}$, denoted by $U(\mathfrak{g})$, is the unital associative algebra over k, satisfying the following universal property:

(i) $\exists$ a Lie alg. hom. $i : \mathfrak{g} \longrightarrow \mathrm{Lie}(U(\mathfrak{g}))$ s.t.

(ii) for every associative, unital alg. B and a Lie alg. hom

$$\sigma : \mathfrak{g} \longrightarrow \mathrm{Lie}(B)$$

$\exists!$ associative alg. hom $\tilde{\sigma} : U(\mathfrak{g}) \to B$ s.t. $\sigma = \tilde{\sigma} \circ i$



σ & i are Lie alg. hom-s

$\tilde{\sigma}$ is an alg. hom.

In other words, $\mathrm{Hom}_{\text{Lie Alg}}(\mathfrak{g}, \mathrm{Lie}(B)) = \mathrm{Hom}_{\text{Assoc. Alg.}}(U(\mathfrak{g}), B)$

for every unital associative alg. B.

In particular there is a direct correspondence

$$\mathfrak{g}\text{-representations} = U(\mathfrak{g})\text{-modules (or representations)}$$

Side Remark.- The fact that we have a natural tensor operation on $\mathfrak{g}$-repns. implies that $U(\mathfrak{g})$ is a "bialgebra":

In fact $U(\mathfrak{g})$ has a structure of a "Hopf algebra".

4. Hopf algebras. - A Hopf algebra H over a field k is a unital, assoc. algebra together with (a) $\Delta : H \to H \otimes H$ alg. hom. (called coproduct or comultiplication)

(b) $\varepsilon : H \to k$ alg. hom. (called co-unit)

(c) $S : H \to H$ alg. anti-hom. (called antipode)

s.t. (i) Δ is coassociative i.e. $(id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta$

$$\begin{array}{ccc} H & \xrightarrow{\Delta} & H \otimes H \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle id\otimes\Delta} \\ H \otimes H & \xrightarrow[\Delta\otimes id]{} & H \otimes H \otimes H \end{array}$$

(ii) $(\varepsilon \otimes id) \circ \Delta = (id \otimes \varepsilon) \circ \Delta = id_H$

$$\begin{array}{ccc} H & \xrightarrow{\Delta} & H \otimes H \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle \varepsilon\otimes id} \\ H & = & k \otimes H \end{array} \qquad \text{and} \qquad \begin{array}{ccc} H & \to & H \otimes H \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id\otimes\varepsilon} \\ H & = & H \otimes k \end{array}$$

(iii) $\mu \circ (S \otimes id) \circ \Delta = \varepsilon(\cdot) \cdot 1_H = \mu \circ (id \otimes S) \circ \Delta$

(here $\mu : H \otimes H \to H$ is the multiplication and $1_H \in H$ is the unit)

$$\begin{array}{ccccccc} & & H \otimes H & \xrightarrow{S\otimes id} & H \otimes H & & \\ & \nearrow{\scriptstyle\Delta} & & & & \searrow{\scriptstyle\mu} & \\ H & \xrightarrow{\varepsilon} & k & \xrightarrow{\lambda \mapsto \lambda \cdot 1_H} & & & H \\ & \searrow & & & & \nearrow{\scriptstyle\mu} & \\ & & H \otimes H & \xrightarrow[id\otimes S]{} & H \otimes H & & \end{array}$$

Roughly, these axioms translate to the following structures on

$\mathcal{C} = \mathrm{Rep}(H) = H$-modules :

I. Tensor product : $\mathrm{Rep}(H) \otimes \times \mathrm{Rep}(H) \to \mathrm{Rep}(H)$

$V, W \in \mathrm{Rep}(H)$, say $\rho_V : H \to \mathrm{End}(V)$ and $\rho_W : H \to \mathrm{End}(W)$ are the action homomorphisms. Then,

$V \otimes W \in \mathrm{Rep}(H)$ with H-action given by

$$H \xrightarrow{\Delta} H \otimes H \longrightarrow \mathrm{End}(V) \otimes \mathrm{End}(W) \subset \mathrm{End}(V \otimes W).$$

More explicitly, if $x \in H$, $v \in V$, $w \in W$, and

$$\Delta(x) = \sum_{i=1}^{m} x_i^{(1)} \otimes x_i^{(2)}$$

then $$x \cdot (v \otimes w) = \sum_{i=1}^{m} \rho_V(x_i^{(1)})(v) \otimes \rho_W(x_i^{(2)})(w)$$

II. Trivial repn. Denote by $\mathbb{1}$ the 1-dim'l repn. arising from the counit $\varepsilon : H \to k = \mathrm{End}(k)$

Axiom (i) : $(V_1 \otimes V_2) \otimes V_3 \to V_1 \otimes (V_2 \otimes V_3)$ is hom. of H-repns. for every $V_1, V_2, V_3 \in \mathrm{Rep}(H)$.

Axiom (ii) : $\mathbb{1} \otimes V \xrightarrow{\sim} V \xleftarrow{\sim} V \otimes \mathbb{1}$ are homs. of H-repns.

III. Duals. Given $V \in \mathrm{Rep}(H)$, consider the H-action on V^* given by $(a \cdot \xi)(v) = \xi(S(a) \cdot v)$ $\forall$ $\xi \in V^*$, $a \in H$, $v \in V$.

Ex. Verify that the axiom (iii) of Hopf algebras translates to the following being hom-s. of H-repns., for every $V \in \mathrm{Rep}(H)$, say V finite-dim'l:

$$V^* \otimes V \xrightarrow[\mathrm{ev}]{} \mathbb{1} \qquad\qquad \mathbb{1} \xrightarrow{\mathrm{coev}} V \otimes V^*$$

$$\xi \otimes v \longmapsto \xi(v) \qquad\qquad 1 \longmapsto \sum_{i=1}^{\dim V} v_i \otimes v^i$$

$\{v_i\}$ a basis of V

$\{v^i\}$ the dual basis of V