Lecture 4

1. Hopf algebras - A *Hopf algebra* over a field k is a k-vector space H, together with the following maps (linear):

$$\mu : H \otimes H \to H \quad , \quad i : k \to H, \quad \Delta : H \to H \otimes H$$

$$\varepsilon : H \to k \quad , \quad S : H \to H$$

called, respectively, *multiplication*, *unit*, *comultiplication*, *counit* and *antipode*. These maps are subject to the following axioms.

(i) $(H, \mu, i(1))$ is a *unital, associative algebra*. This means:

$$\begin{array}{ccc} H \otimes H \otimes H & \xrightarrow{\mu \otimes id} & H \otimes H \\ {\scriptstyle id \otimes \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ H \otimes H & \xrightarrow[\mu]{} & H \end{array}$$

$$\begin{array}{ccccc} k \otimes H & \xrightarrow{i \otimes id} & H \otimes H & \xleftarrow{id \otimes i} & H \otimes k \\ & \searrow^{\cong *} & \downarrow{\scriptstyle \mu} & \swarrow_{\cong *} & \\ & & H & & \end{array}$$

these diagrams commute

(ii) (H, Δ, ε) is a *counital, coassociative, coalgebra* - i.e.,

- Δ is coassociative i.e. $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$

$$\begin{array}{ccc} H & \xrightarrow{\Delta} & H \otimes H \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \Delta \otimes id} \\ H \otimes H & \xrightarrow[id \otimes \Delta]{} & H \otimes H \otimes H \end{array} \qquad \text{commutes}$$

- ε is counit - i.e. an alg hom. s.t.

$$\begin{array}{ccccc} & \cong \nearrow & H & \nwarrow \cong & \\ & & \downarrow{\scriptstyle \Delta} & & \text{commutes} \\ k \otimes H & \xleftarrow[\varepsilon \otimes id]{} & H \otimes H & \xrightarrow[id \otimes \varepsilon]{} & H \otimes k \end{array}$$

* for a k-vector space V

$$k \otimes V \xrightarrow{\cong} V$$
$$(\lambda \otimes v) \mapsto \lambda v$$

$$V \otimes k \xrightarrow{\cong} V$$
$$v \otimes \lambda \mapsto \lambda v$$

(iii) (compatibility between (i) and (ii))

$\Delta : H \to H \otimes H$ is an algebra hom - where multiplication on $H \otimes H$ is given by $(a \otimes b) \cdot (c \otimes d) = ac \otimes bd$; $\forall\, a, b, c, d \in H$.

In terms of commutative diagrams :

$$\begin{array}{ccccc} H \otimes H & \xrightarrow{\mu} & H & \xrightarrow{\Delta} & H \otimes H \\ & \searrow^{\Delta \otimes \Delta} & & & \uparrow{\scriptstyle \mu \otimes \mu} \\ & & H \otimes H \otimes H \otimes H & \xrightarrow{(23)} & H \otimes H \otimes H \otimes H \end{array}$$

$$a_1 \otimes a_2 \otimes a_3 \otimes a_4 \longmapsto a_1 \otimes a_3 \otimes a_2 \otimes a_4$$

Ex. This axiom is equivalent to requiring μ to be a coalgebra homomorphism.

(iv) (Antipode axiom) - $\mu \circ (S \otimes id) \circ \Delta = i \circ \varepsilon = \mu \circ (id \otimes S) \circ \Delta$

S is an alg. anti-hom and

$$S(ab) = S(b)\, S(a)$$

$$\begin{array}{ccccccc} H & \xrightarrow{\Delta} & H \otimes H & \xrightarrow[\text{or}]{\substack{S \otimes id \\ id \otimes S}} & H \otimes H & \xrightarrow{\mu} & H \\ & \searrow^{\varepsilon} & & & & \nearrow_{i} & \\ & & k & & & & \end{array}$$

Sweedler's notation - for $x \in H$, $\Delta(x) \in H \otimes H$ is a finite sum of pure tensors, written usually as $\Delta(x) = \sum_{i=1}^{n} x_i^{(1)} \otimes x_i^{(2)}$.

In Sweedler's notation, one often suppresses $\sum_{i=1}^{n}$ and just writes $\Delta(x) = x^{(1)} \otimes x^{(2)}$ – summation being assumed.

2. Structure on $\mathrm{Rep}(H)$.-

Axioms (ii) & (iii) gives a bifunctor

$$\otimes : \mathrm{Rep}(H) \times \mathrm{Rep}(H) \longrightarrow \mathrm{Rep}(H)$$ as follows:

If $\rho_V : H \to \mathrm{End}(V)$ and $\rho_W : H \to \mathrm{End}(W)$ are two representations of H, then we define H-action on $V \otimes W$ by:

$$H \xrightarrow{\Delta} H \otimes H \xrightarrow{\rho_V \otimes \rho_W} \mathrm{End}(V) \otimes \mathrm{End}(W) \subset \mathrm{End}(V \otimes W)$$

More explicitly, for all $x \in H$, $v \in V$ and $w \in W$,

$$x \cdot (v \otimes w) = \sum_{i=1}^{n} \rho_V(x_i^{(1)})(v) \otimes \rho_W(x_i^{(2)})(w)$$

if $\Delta(x) = \sum_{i=1}^{n} x_i^{(1)} \otimes x_i^{(2)}$.

By coassociativity of Δ, the natural isomorphism

$$(V_1 \otimes V_2) \otimes V_3 \longrightarrow V_1 \otimes (V_2 \otimes V_3)$$

is a homomorphism of H-representations.

Also Axiom (ii) gives an object (unit for $\otimes$), denoted by $\mathbb{1}$, called the trivial representation – which is 1-dim'l k-vector space, with H-action coming from counit $\varepsilon : H \to \mathrm{End}_k(k) = k$

i.e. $x \cdot z = \varepsilon(x) z$, $\forall\, x \in H$, $z \in k$.

Finally the antipode S allows us to define $\mathcal{H}$-action on the dual vector space $\mathrm{Hom}_k(V, k)$ for any $\mathcal{H}$-repn. V; called the left dual, denoted by ${}^*V$, as follows:

$$(x \cdot \xi)(v) = \xi(S(x) \cdot v), \quad \forall\ x \in \mathcal{H},\ \xi \in {}^*V,\ v \in V.$$

The antipode axiom translates to the fact that the natural maps (for V a finite dim'l $\mathcal{H}$-repn.)

$$V^* \otimes V \xrightarrow{\text{ev}} \mathbb{1} \qquad \text{and} \qquad \mathbb{1} \xrightarrow{\text{coev}} V \otimes V^*$$

$$\xi \otimes v \longmapsto \xi(v) \qquad 1 \longmapsto \sum_{i=1}^{\dim(V)} v_i \otimes v^i$$

are $\mathcal{H}$-intertwiners.

$\{v_i\}$ a basis of V

$\{v^i\}$ the dual basis of ${}^*V$

$(v^i(v_j) = \delta_{ij})$

3. Examples.- (a) Group algebra of a finite group G, over a field k,

is defined as $k[G] = k$-vector space over basis $\{\delta_g\}_{g \in G}$

with multiplication given by $\delta_g * \delta_h = \delta_{gh}$. δ_e = unit

Hopf algebra structure -

$$\Delta : k[G] \longrightarrow k[G] \otimes k[G]$$
$$\delta_g \longmapsto \delta_g \otimes \delta_g$$

$$S : k[G] \to k[G] \qquad \varepsilon : k[G] \to k$$
$$\delta_g \mapsto \delta_{g^{-1}} \qquad \delta_g \mapsto 1 \ (\forall\ g \in G)$$

(b) Tensor algebra of a vector space V over k. - $T^\cdot(V)$ is defined as $\mathbb{Z}_{\geq 0}$ - graded algebra $T^\cdot(V) = \bigoplus_{n=0}^{\infty} T^n(V)$ where

$T^0(V) = k$ and $T^n(V) = \underbrace{V \otimes \cdots \otimes V}_{n\text{-fold}}$. The multiplication is given by concatenation of tensors

$$T^n(V) \otimes T^m(V) \longrightarrow T^{n+m}(V)$$

$$(v_1 \otimes \cdots \otimes v_n) \otimes (u_1 \otimes \cdots \otimes u_m) \longmapsto v_1 \otimes \cdots \otimes v_n \otimes u_1 \otimes \cdots \otimes u_m$$

Hopf algebra structure - $\Delta : T^\cdot(V) \to T^\cdot(V) \otimes T^\cdot(V)$ is the unique algebra hom. extending

$$\Delta(v) = v \otimes 1 + 1 \otimes v \quad , \quad \forall\, v \in V$$

Note - Universal property of $T^\cdot$: Vect. Spaces $\to$ (unital) Assoc. Alg's

$$\operatorname{Hom}_{\text{Alg}}(T^\cdot(V), A) = \operatorname{Hom}_{k\text{-v.s.}}(V, A)$$ for any assoc. alg. A

Take $A = T^\cdot(V) \otimes T^\cdot(V)$ and consider the linear map

$$v \longmapsto v \otimes 1 + 1 \otimes v \quad , \quad \forall\, v \in V.$$

Similarly define $\varepsilon : T^\cdot(V) \to k$, $v \mapsto 0$ $\forall\, v \in V$, alg. hom

$S : T^\cdot(V) \to T^\cdot(V)$, $v \mapsto -v$, alg. anti-hom

4. Hopf structure on $U(\mathfrak{g})$ - Recall that $U(\mathfrak{g})$ is the unique unital, associative algebra (over k) satisfying the universal property

$$\underset{\text{Alg}}{\text{Hom}}\,(U(\mathfrak{g}),\, A) \;=\; \underset{\text{LieAlg}}{\text{Hom}}\,(\mathfrak{g},\, \text{Lie}(A))$$

Comultiplication - The linear map $i:\mathfrak{g} \to U(\mathfrak{g})\otimes U(\mathfrak{g})$ is a homomorphism of Lie algebras: $x \mapsto x\otimes 1 + 1\otimes x$

Proof. Multiplication on $U(\mathfrak{g})^{\otimes 2}$ is given componentwise - so $[i(x), i(y)] = [x\otimes 1+1\otimes x, y\otimes 1+1\otimes y]$

$$= [x,y]\otimes 1 + 1\otimes [x,y] = i([x,y]).$$

So, by the universal property, there is a unique alg. hom.

$$\Delta : U(\mathfrak{g}) \to U(\mathfrak{g})\otimes U(\mathfrak{g}) \text{ extending } x\mapsto x\otimes 1+1\otimes x \quad (\forall\, x\in\mathfrak{g}).$$

Counit - (details left as an easy exercise)

$$\varepsilon : U(\mathfrak{g}) \to k \text{ unique alg. hom. } x\mapsto 0 \quad (\forall\, x\in\mathfrak{g}).$$

Antipode - $S : U(\mathfrak{g}) \to U(\mathfrak{g})$ unique alg. anti-hom determined by $S(x) = -x \quad \forall\, x\in\mathfrak{g}$.

Ex. - Verify the axioms of Hopf algebra