Lecture 5

1. Construction of $U(\mathfrak{g})$ - Let $\mathfrak{g}$ be a Lie algebra over a field k. Recall that $T^\cdot(\mathfrak{g})$ denotes the tensor algebra of $\mathfrak{g}$, which is graded over $\mathbb{Z}_{\geq 0}$

$$T^\cdot(\mathfrak{g}) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} T^n(\mathfrak{g}), \quad \text{where}$$

$$T^0(\mathfrak{g}) = k \quad \text{and} \quad T^n(\mathfrak{g}) = \underbrace{\mathfrak{g} \otimes \cdots \otimes \mathfrak{g}}_{n\text{-fold}} = \mathfrak{g}^{\otimes n}.$$

We will denote by $*$ the multiplication on $T^\cdot(\mathfrak{g})$

$$* : T^n(\mathfrak{g}) \otimes T^m(\mathfrak{g}) \to T^{n+m}(\mathfrak{g}).$$

$$\xi * \eta = \xi \otimes \eta$$

Let $\mathcal{I} \subset T^\cdot(\mathfrak{g})$ be the two-sided ideal generated by

$$\{ x*y - y*x - [x,y] : x, y \in \mathfrak{g} \}$$

Note: $\mathcal{I}$ is <u>not</u> a graded ideal $\quad x*y,\ y*x \in T^2(\mathfrak{g})$

$$[x,y] \in T^1(\mathfrak{g}) = \mathfrak{g}.$$

$$\boxed{U(\mathfrak{g}) := T^\cdot(\mathfrak{g}) / \mathcal{I}}$$

The homomorphism $i : \mathfrak{g} \to \mathrm{Lie}(U(\mathfrak{g}))$ is constructed as follows:

$$\mathfrak{g} = T^1(\mathfrak{g}) \hookrightarrow T^\cdot(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g})$$

The fact that i is a Lie algebra homomorphism follows immediately from the definition of the ideal $\mathcal{I}$.

Prop. [universal property of $\mathcal{U}(\mathfrak{g})$] - Let A be an arbitrary unital, associative algebra over k and $\sigma : \mathfrak{g} \to \mathrm{Lie}(A)$ be a hom. of Lie algebras. Then, $\exists!$ hom. of (unital) assoc. algebras

$$\tilde{\sigma} : \mathcal{U}(\mathfrak{g}) \to A \quad \text{such that} \quad \tilde{\sigma}(i(x)) = \sigma(x) \quad \forall\, x \in \mathfrak{g}.$$

Pf. - By the universal property of tensor algebra, the linear map $\sigma : \mathfrak{g} \to A$ extends to a unique alg. hom (unital)

$$\tilde{\sigma} : T^{\cdot}(\mathfrak{g}) \longrightarrow A$$
$$\tilde{\sigma}(x) = \sigma(x) \quad \forall\, x \in T^1(\mathfrak{g}) = \mathfrak{g}.$$

Since σ is a hom. of Lie algebras, we get that

$$x * y - y * x - [x,y] \in \mathrm{Ker}(\tilde{\sigma}) \;;\; \forall\, x, y \in \mathfrak{g}$$

i.e. $\tilde{\sigma}$ descends to (factors through) $T^{\cdot}(\mathfrak{g}) \big/ _{\mathcal{I}} = \mathcal{U}(\mathfrak{g})$.

Uniqueness follows since $\{i(x) : x \in \mathfrak{g}\}$ generate $\mathcal{U}(\mathfrak{g})$ as an algebra. □

Example - If $\mathfrak{g}$ is abelian (i.e. $[x,y] = 0 \;\forall\, x, y \in \mathfrak{g}$), then

$$\mathcal{U}(\mathfrak{g}) = T^{\cdot}(\mathfrak{g}) \Big/ \begin{array}{l}\text{two-sided ideal} \\ \text{generated by } x*y - y*x \,;\; x, y \in \mathfrak{g}\end{array}$$

is, by definition, the symmetric algebra $\mathrm{Sym}^{\cdot}(\mathfrak{g})$.

2. Filterations and associated graded - Let A be a unital, associative algebra over k. A filteration $F_\bullet(A) = \{F_n(A)\}_{n \in \mathbb{Z}_{\geq 0}}$ on A is a sequence of subspaces of A

$$F_0(A) = k \subset F_1(A) \subset F_2(A) \subset \cdots$$

so that $A = \bigcup_{n=0}^{\infty} F_n(A)$ and $\begin{matrix} a \in F_n(A) \\ b \in F_m(A) \end{matrix}$ implies $a \cdot b \in F_{n+m}(A)$.

Recall that a graded algebra (over $\mathbb{Z}_{\geq 0}$) B (over k) is a unital, associative algebra, together with subspaces $B_n \subset B$ $(n \in \mathbb{Z}_{\geq 0})$ s.t.

$$B = \bigoplus_{n=0}^{\infty} B_n \quad \text{and} \quad \begin{matrix} a \in B_n \\ b \in B_m \end{matrix} \text{ implies } a \cdot b \in B_{n+m}$$

as vector spaces

If A, $F_\bullet(A)$ is a filtered algebra, we define its associated graded

$$\mathrm{gr}_F(A) := \bigoplus_{n=0}^{\infty} F_n(A)/F_{n-1}(A) \qquad (F_{-1}(A) = \{0\} \text{ by convention})$$

with multiplication given by:

if $a \in \mathrm{gr}_n(A) = F_n/F_{n-1}$, $b \in \mathrm{gr}_m(A) = F_m/F_{m-1}$, then choose representatives $\tilde{a} \in F_n$, $\tilde{b} \in F_m$ of a & b resp.

and set $a \cdot b := \tilde{a} \cdot \tilde{b} \bmod F_{n+m-1} \in F_{n+m}/F_{n+m-1} = \mathrm{gr}_{n+m}(A)$

Easy exercise - This definition is unambiguous.

3. Filtration on $\mathcal{U}(\mathfrak{g})$ - Let $\{F_n(\mathcal{U}(\mathfrak{g}))\}_{n\in\mathbb{Z}_{\geq 0}}$ be defined by

$$\bigoplus_{j=0}^{n} T^j(\mathfrak{g}) \hookrightarrow T^{\cdot}(\mathfrak{g}) \xrightarrow{\pi} \mathcal{U}(\mathfrak{g})$$

$$F_n(\mathcal{U}(\mathfrak{g})) := \pi\left(\bigoplus_{j=0}^{n} T^j(\mathfrak{g})\right) \qquad F_0(\mathcal{U}(\mathfrak{g})) = k$$

Remark. - If $\{x_\lambda\}_{\lambda\in\Lambda}$ is a basis of $\mathfrak{g}$, then $T^{\cdot}(\mathfrak{g})$ has a basis consisting of monomials in $\{x_\lambda\}$ - non commuting:

$$\{x_{\lambda_1} * \cdots * x_{\lambda_n} : \lambda_1, \ldots, \lambda_n \in \Lambda\} \text{ basis of } T^n(\mathfrak{g}).$$

Thus $F_n(\mathcal{U}(\mathfrak{g}))$ is spanned by such monomials of length $\leq n$.

Lemma. - Let $a_1, \ldots, a_n \in \mathfrak{g}$ and $\sigma \in S_n$ (permutation).

Then $a_1 * \cdots * a_n - a_{\sigma(1)} * \cdots * a_{\sigma(n)} \in F_{n-1}(\mathcal{U}(\mathfrak{g}))$

Proof. - The statement is vacuous for $n = 1$. For $n \geq 2$, S_n is generated by simple transpositions $\sigma_i = (i\ i+1)$, $1 \leq i \leq n-1$

and for $\sigma = \sigma_i$, the expression in the lemma becomes

$$a_1 \cdots a_{i-1} (a_i a_{i+1} - a_{i+1} a_i) a_{i+2} \cdots a_n$$

$$= a_1 \cdots a_{i-1} [a_i, a_{i+1}] a_{i+2} \cdots a_n \in F_{n-1}(\mathcal{U}(\mathfrak{g}))$$

as claimed

□

Cor. 1- The associated graded $gr_F(\mathcal{U}(\mathfrak{g}))$ is commutative.

Cor. 2. For each $n \geq 0$, we have a linear map ω_n

$$\begin{array}{ccc} Sym^n(\mathfrak{g}) & \twoheadleftarrow & T^n(\mathfrak{g}) \\ \omega_n \,\vdots\, \downarrow & & \downarrow \\ F_n(\mathcal{U}(\mathfrak{g}))/F_{n-1}(\mathcal{U}(\mathfrak{g})) & \twoheadleftarrow & F_n(\mathcal{U}(\mathfrak{g})) \end{array}$$

giving rise to a surjective alg. hom. $\omega : Sym^{\cdot}(\mathfrak{g}) \to gr_F(\mathcal{U}(\mathfrak{g}))$.

4. Poincaré - Birkhoff - Witt theorem. - The map ω is an isomorphism.

Remarks - (a) If $\{x_\lambda\}_{\lambda \in \Lambda}$ is a basis of $\mathfrak{g}$, and $<$ a fixed total order on the index set Λ, then the following set is a basis of $\mathcal{U}(\mathfrak{g})$:

$$\left\{ x_{\lambda_1} \cdots x_{\lambda_r} : \begin{array}{l} r \geq 0 \\ \lambda_1 \leq \lambda_2 \cdots \leq \lambda_r \text{ in } \Lambda \end{array} \right\}$$

(empty word, for $r = 0$, is assumed to be 1)

(b) The statement of this theorem fails if k is replaced by an arbitrary commutative ring.