Lecture 6

1. Let $\mathfrak{g}$ be a Lie algebra over a field k. Recall that we defined the universal enveloping algebra $U(\mathfrak{g})$ as the quotient of the (graded) tensor algebra $T^\cdot(\mathfrak{g})$ by the two-sided ideal generated by the following set of elements: $x * y - y * x - [x,y]$, $\forall\, x, y \in \mathfrak{g}$. Here, $*$ denotes the multiplication of $T^\cdot(\mathfrak{g})$.

By the PBW theorem, if $\{x_i\}_{i\in I}$ is a basis of $\mathfrak{g}$, and $<$ is a total order on the indexing set I, then the set of ordered monomials

$$\mathcal{X} = \left\{ x_{i_1} \cdots x_{i_\ell} : \begin{array}{c} \ell \in \mathbb{Z}_{\geq 0} \\ i_1 \leq \ldots \leq i_\ell \text{ in } I \end{array} \right\} \text{ forms a basis of } U(\mathfrak{g}).$$

(empty monomial $= 1$ $\ell = 0$)

Corollaries. - (proofs are left as exercise)

(i) The canonical map $i : \mathfrak{g} \to U(\mathfrak{g})$ is injective.

(ii) Let $F_\cdot$ be the canonical filtration on $U(\mathfrak{g})$.
Then $\mathrm{gr}_F(U(\mathfrak{g})) \simeq \mathrm{Sym}^\cdot(\mathfrak{g})$ as graded algebras.

(iii) $U(\mathfrak{g})$ has no non-zero zero divisors.

(iv) Recall that for a Hopf algebra H, we say $x \in H$ is primitive if $\Delta(x) = x \otimes 1 + 1 \otimes x$. Let $\mathrm{Prim}(H)$ denote the (Lie) subalgebra of primitive elements of H.

If $\mathrm{Char}(k) = 0$, $\mathrm{Prim}(U(\mathfrak{g})) = \mathfrak{g}$.

2. Free Lie algebra over a vector space V - denoted by $\mathcal{L}_V$, is

defined via the following universal property:

(i) There is a linear map $i : V \to \mathcal{L}_V$.

(ii) For a Lie algebra $\mathfrak{g}$ and a linear map $\varphi : V \to \mathfrak{g}$, there is a unique Lie alg. hom. $\widetilde{\varphi} : \mathcal{L}_V \to \mathfrak{g}$ s.t.

$$\widetilde{\varphi}(i(v)) = \varphi(v) \quad \forall\, v \in V.$$

Lie Alg's $\rightleftarrows$ Vector Spaces (Forgetful: Lie Alg's $\to$ Vector Spaces; $\mathcal{L}$: Vector Spaces $\to$ Lie Alg's)

$$\underset{\text{LieAlg}}{\mathrm{Hom}}(\mathcal{L}_V, \mathfrak{g}) = \underset{\text{v.s.}}{\mathrm{Hom}}(V, \mathfrak{g})$$

Construction I. - Choose a basis $\{v_i\}_{i \in I}$ of V and let W_V be the vector space (over k) on basis given by all non-empty words, with complete bracketing, on alphabet $\{v_i\}_{i \in I}$

e.g. if $\dim V = 2$ with basis $\{x, y\}$, W_V consists of elements of the form

x: (xx), $(x(xy))$, $((xx)y)$

(xy), $(x(yx))$, $((xy)x)$ ...

y: (yx), $(y(xx))$, $((yx)x)$

(y,y)

Define the bilinear operation (concatenation)

$$\mu : W_V \times W_V \to W_V$$
$$w_1, w_2 \mapsto ((w_1)(w_2))$$

[graded by $\deg(v_i) = 1$ $\forall\, i \in I$]

Now take $\mathcal{L}_V$ to be the quotient of W_V by the 2-sided

ideal (w.r.t. μ) generated by following elements

~~$(v_i v_i)$~~ $\quad i \in I$

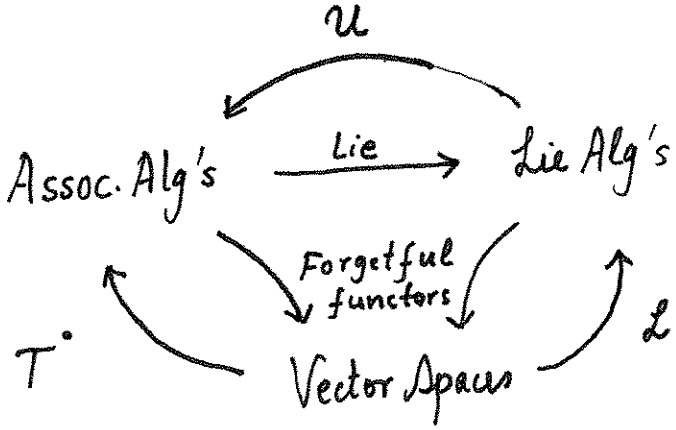
$$(v_i (v_j v_k)) + (v_j (v_k v_i)) + (v_k (v_i v_j)) \quad , \; i,j,k \in I.$$

Note : $\mathcal{L}_V$ is again graded $\mathcal{L}_V = \bigoplus_{n=1}^{\infty} \mathcal{L}_V[n]$,

$\mathcal{L}_V[1] = V$, $\mathcal{L}_V[2]$ can be identified with $V \wedge V$, ...

If $\dim V > 1$, $\mathcal{L}_V$ is infinite dimensional, but (assuming $\dim V = N < \infty$), each graded piece $\mathcal{L}_V[n]$ is finite-dim'l.

Construction II. -



By universality properties $\mathcal{U}(\mathcal{L}_V) \simeq T^\cdot(V)$

Thus, in characteristic zero, $\mathcal{L}_V \simeq \mathrm{Prim}(T^\cdot(V))$, i.e. the smallest subspace containing V which is stable under the commutator $ab - ba$ in $T^\cdot(V)$.

3. Lie algebra given by generators and relations - A Lie algebra $\mathfrak{g}$ on a set of generators $\{x_i\}_{i\in I}$ satisfying a set of relations $R \subset \mathcal{L}_V$ where V is k-vector space on basis $\{x_i\}_{i\in I}$ is, by definition, the quotient of $\mathcal{L}_V$ by the (Lie) ideal generated by R. Thus,

$$\operatorname{Hom}_{\text{LieAlg}}(\mathfrak{g}, \mathfrak{h}) = \left\{ \varphi : V \to \mathfrak{h} \text{ linear s.t. } R \subset \operatorname{Ker}(\tilde{\varphi} : \mathcal{L}_V \to \mathfrak{h}) \right\}$$

Equivalently, $\mathcal{U}(\mathfrak{g}) = k\langle x_i : i\in I\rangle \Big/ i(R)$

$(\, i((ab)) = ab - ba \;\; \forall\, a, b \in V = k\text{-v.s. on } \{x_i\}\,)$

e.g. $\mathfrak{g}$ = Lie algebra generated by $\{x, y\}$ subject to relations $[x,[x,y]] = 0 = [y,[y,x]]$

$$\mathcal{U}(\mathfrak{g}) = k\langle x, y\rangle \Big/ \left\langle \begin{matrix} x^2y - 2xyx + yx^2 \\ y^2x - 2yxy + xy^2 \end{matrix} \right\rangle \text{ two-sided ideal.}$$

Note: $\mathfrak{g}$ is 3-dim'l as k-vector space, on basis $x, y, [x,y]$

So $\mathcal{U}(\mathfrak{g})$ admits the following basis, by PBW thm.

$$\mathcal{U}(\mathfrak{g}) = k\text{-span of } \left\{ x^a y^b [x,y]^c : a, b, c \in \mathbb{Z}_{\geq 0} \right\}$$