Lecture 7

1. Let $\mathfrak{g}$ be a Lie algebra over a field k. Let $\mathfrak{a}_1, \mathfrak{a}_2 \subset \mathfrak{g}$ be two ideals. We define $[\mathfrak{a}_1, \mathfrak{a}_2]$ to be the subspace of $\mathfrak{g}$ spanned by

$$\{ [x_1, x_2] : x_1 \in \mathfrak{a}_1, \ x_2 \in \mathfrak{a}_2 \}.$$

Ex. - $[\mathfrak{a}_1, \mathfrak{a}_2] \subset \mathfrak{g}$ is again an ideal.

Definition - We say $\mathfrak{g}$ is solvable if $\mathcal{D}^N(\mathfrak{g}) = \{0\}$ for $N >> 0$, where $\mathcal{D}^0(\mathfrak{g}) = \mathfrak{g} \supset \mathcal{D}^1(\mathfrak{g}) \supset \dots$ is defined by

$$\mathcal{D}^{n+1}(\mathfrak{g}) = [\mathcal{D}^n(\mathfrak{g}), \mathcal{D}^n(\mathfrak{g})]$$

We say $\mathfrak{g}$ is nilpotent if $\mathcal{D}_N(\mathfrak{g}) = 0$ for $N >> 0$, where the descending chain $\mathcal{D}_0(\mathfrak{g}) = \mathfrak{g} \supset \mathcal{D}_1(\mathfrak{g}) \supset \dots$ is defined by

$$\mathcal{D}_{n+1}(\mathfrak{g}) = [\mathfrak{g}, \mathcal{D}_n(\mathfrak{g})]$$

Main example. (1) $\mathfrak{b}_n(k)$ = Lie algebra of upper triangular $n \times n$ matrices (entries from k) is solvable. $\mathfrak{u}_n(k)$ = Lie alg. of strict upper triangular matrices is nilpotent.

$(n=2)$
$$\left[\begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ 0 & d_2 \end{bmatrix} \right] = \begin{bmatrix} 0 & a_1b_2 - a_2b_1 + b_1d_2 - b_2d_1 \\ 0 & 0 \end{bmatrix}$$

$$\mathcal{D}^0(\mathfrak{b}_2(k)) = \mathfrak{b}_2(k) \supset \mathcal{D}^1(\mathfrak{b}_2(k)) = \mathfrak{u}_2(k) \supset \mathcal{D}^2(\mathfrak{b}_2(k)) = \{0\}$$

Remarks.- Every abelian Lie algebra is nilpotent. Every nilpotent Lie algebra is solvable (show that $\mathcal{D}^n(\mathfrak{g}) \subset \mathcal{D}_n(\mathfrak{g})$ for all n).

2. Proposition - (a) $\mathfrak{g}$ is solvable if, and only if there is a sequence of ideals $\mathfrak{g} = I_0 \supset I_1 \supset \cdots \supset I_m = \{0\}$ s.t.

$$I_j / I_{j+1} \text{ is abelian } \forall\ 0 \le j \le m-1.$$

(b) $\mathfrak{g}$ is nilpotent if and only if there is a sequence of ideals

$$\mathfrak{g} = J_0 \supset J_1 \supset \cdots \supset J_n = \{0\} \text{ s.t.}$$

$$[\mathfrak{g}, J_j] \subset J_{j+1} \quad \forall\ 0 \le j \le n-1.$$

Proof.- (a) ($\Rightarrow$) Take $I_j = \mathcal{D}^j(\mathfrak{g})$, then $I_j / I_{j+1} = I_j / [I_j, I_j]$ is abelian $\forall j \ge 0$.

($\Leftarrow$) We claim, by induction on j, that $\mathcal{D}^j(\mathfrak{g}) \subset I_j$. The base case is clear. For the induction step,

$$\mathcal{D}^j(\mathfrak{g}) \subset I_j \text{ implies } \mathcal{D}^{j+1}(\mathfrak{g}) = [\mathcal{D}^j(\mathfrak{g}), \mathcal{D}^j(\mathfrak{g})] \subset [I_j, I_j]$$

Since I_j / I_{j+1} is abelian, by Problem 2 of Set 1,

$$[I_j, I_j] \subset I_{j+1} \text{ and hence } \mathcal{D}^{j+1}(\mathfrak{g}) \subset I_{j+1}.$$

Finally $\mathcal{D}^m(\mathfrak{g}) \subset I_m = \{0\}$ implies that $\mathfrak{g}$ is solvable.

(b) the proof is similar and hence omitted. □

3. Proposition.- Let $I \subset \mathfrak{g}$ be an ideal.

(a) $\mathfrak{g}$ is solvable $\Leftrightarrow$ I and $\mathfrak{g}/I$ are solvable

(b) Assume $I \subset Z(\mathfrak{g})$ center of $\mathfrak{g}$. Then $\mathfrak{g}$ is nilpotent $\Leftrightarrow$ $\mathfrak{g}/I$ is nilpotent.

Proof.- (a) Let $\pi : \mathfrak{g} \to \mathfrak{g}/I$ be the natural surjection.

($\Rightarrow$): $\mathcal{D}^n(I) \subset \mathcal{D}^n(\mathfrak{g})$ and $\pi(\mathcal{D}^n(\mathfrak{g})) \twoheadrightarrow \mathcal{D}^n(\mathfrak{g}/I)$. So, if $\mathcal{D}^N(\mathfrak{g}) = 0$ then $\mathcal{D}^N(I)$ and $\mathcal{D}^N(\mathfrak{g}/I)$ are also $\{0\}$.

($\Leftarrow$) Let $\overline{\mathfrak{g}} = \overline{\mathfrak{g}}_0 \supset \overline{\mathfrak{g}}_1 \supset \cdots \supset \overline{\mathfrak{g}}_e = \{0\}$ and

$I = I_0 \supset I_1 \supset \cdots \supset I_t = \{0\}$ be the sequences of ideals satisfying (a) of Prop. 2 above. Then the following sequence of ideals in $\mathfrak{g}$ again satisfies this condition (check!)

$$\mathfrak{g} = \bar{\pi}^{-1}(\overline{\mathfrak{g}}) \supset \bar{\pi}^{-1}(\overline{\mathfrak{g}}_1) \supset \cdots \supset \bar{\pi}^{-1}(\overline{\mathfrak{g}}_e) = I \supset I_1 \supset \cdots \supset I_t$$

Proof of (b) is similar, hence omitted. □

4. Remark.- If $\mathfrak{g}$ is nilpotent and

$\mathfrak{g} = J_0 \supset \cdots \supset J_n = \{0\}$ is a sequence of ideals

so that $[\mathfrak{g}, J_j] \subset J_{j+1}$, as in Prop. 2 above. then

for every $x, y \in \mathfrak{g}$, $\operatorname{ad}(x)^{\ell} \cdot y = \underbrace{[x,[x,\dots,[x,y]\cdots]]}_{\ell\text{-fold}} \in J_\ell$.

Hence $\operatorname{ad}(x)^n = 0 \ \forall\, x \in \mathfrak{g}$ showing that each $\operatorname{ad}(x)$ is a nilpotent operator. The converse is also true, for finite-dim'l $\mathfrak{g}$, as we'll see after Engel's theorem.

5. Lemma. Let $\mathfrak{g}$ be a Lie algebra of k, which has a codim 1 ideal $\mathfrak{a} \subset \mathfrak{g}$. Thus $\mathfrak{g} = \mathfrak{a} \oplus kx$ as vector spaces and $[x, \mathfrak{a}] \subset \mathfrak{a}$. (So, $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{a}$.)

Assume V is a finite-dim'l representation of $\mathfrak{g}$ and there is a non-zero $v \in V$ and a linear form $\lambda : \mathfrak{a} \to k$ s.t.

$$a \cdot v = \lambda(a)\, v \qquad \forall\, a \in \mathfrak{a}.$$

(i) W = Span of $\{x^n \cdot v : n \geq 0\} \subset V$ is a subrepn and for every $a \in \mathfrak{a}$, $a - \lambda(a)\cdot \mathrm{Id}$ acts nilpotently on W

(ii) If $\operatorname{Char}(k) = 0$, then $a \cdot w = \lambda(a)\, w \ \forall\, w \in W$ and $\lambda|_{[\mathfrak{g},\mathfrak{g}]} = 0$.

Proof. - Let $\{v, x\cdot v, x^2 \cdot v, \dots, x^{\ell-1}\cdot v\}$ be a basis of W.

Using (*) $a \cdot (x^n \cdot v) = x \cdot (a \cdot x^{n-1} \cdot v) + [a,x] \cdot x^{n-1} \cdot v$

$\forall\, a \in \mathfrak{a}$, $n \in \mathbb{Z}_{\geq 1}$

and $a \cdot v = \lambda(a) \cdot v$, we can show - by induction

on n, that $a \cdot (x^n \cdot v) = \lambda(a)(x^n \cdot v)$, $\forall a \in \mathfrak{a}$

modulo the subspace spanned by $\{v, \dots, x^{n-1} \cdot v\}$

Hence W is a subrepresentation, and $a - \lambda(a) \cdot Id$ is nilpotent.

(ii) Assume $\mathrm{char}(k) = 0$. For every $a \in \mathfrak{a}$,

$$\mathrm{Trace}(a \text{ acting on } W) = \ell \cdot \lambda(a).$$

However, $\mathrm{Tr}([x_1, x_2]) = 0$ for any two linear operations on W,

hence $\forall z \in [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{a}$, $\ell \cdot \lambda(z) = 0 \Rightarrow \lambda(z) = 0$. (char 0)

Thus, again by induction on n, and using (*), we conclude that $a \cdot (x^n \cdot v) = \lambda(a)\, x^n \cdot v \quad \forall n \geq 0$.

(e.g. $a \cdot (x \cdot v) = x \cdot (a \cdot v) + [a, x] \cdot v$

$= \lambda(a)\, x \cdot v + \lambda([a, x])\, v$

$\lambda([a,x]) = 0$ since $\lambda|_{[\mathfrak{g},\mathfrak{g}]} = 0$.

□

6. Lie's theorem - [k : algebraically closed of char. 0]

Let $\mathfrak{g}$ be a finite-dim'l solvable Lie algebra and V be a non-zero finite-dimensional $\mathfrak{g}$-representation. Then there exists $0 \neq v \in V$ and $\lambda : \mathfrak{g} \to k$ linear form s.t. $x \cdot v = \lambda(x) v \quad \forall x \in \mathfrak{g}$.

Proof.- (assume $\mathfrak{g} \neq \{0\}$, otherwise the statement is trivial)

We will argue by induction on $\dim(\mathfrak{g})$. If $\dim(\mathfrak{g}) = 1$, then any eigenvector for a basis element $0 \neq x \in \mathfrak{g}$ acting on V fulfils the condition in the theorem (note: we need $k = \bar{k}$ and $\dim V < \infty$ to ensure eigenvalues and eigenvectors exist).

If $\dim(\mathfrak{g}) > 1$, let $\mathfrak{h} \subsetneq \mathfrak{g}$ be any (proper) codim 1 subspace so that $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{h}$. This is possible since $\mathfrak{g}$ is solvable hence $[\mathfrak{g},\mathfrak{g}] \subsetneq \mathfrak{g}$. Since $\mathfrak{g}/_{[\mathfrak{g},\mathfrak{g}]}$ is abelian, $\mathfrak{h}$ is an ideal in $\mathfrak{g}$.

Now $\dim(\mathfrak{h}) = \dim(\mathfrak{g}) - 1$ and $\mathfrak{h}$ is again solvable by Prop 4(a) above, so, by induction hypothesis, we can find $\lambda : \mathfrak{h} \to k$ linear form and $0 \neq v_0 \in V$ s.t. $h \cdot v_0 = \lambda(h) v_0$, $\forall h \in \mathfrak{h}$.

Pick $x \in \mathfrak{g} \setminus \mathfrak{h}$ so that $\mathfrak{g} = \mathfrak{h} \oplus k \cdot x$ as vector space. Using Lemma 5 above;

$W = k\text{-span of } \{v_0, x \cdot v_0, x^2 \cdot v_0, \cdots\} \subset V$ is a (non-zero) $\mathfrak{g}$-subrepn

and $h \cdot w = \lambda(h) w \quad \forall h \in \mathfrak{h}, w \in W$

$\lambda|_{[\mathfrak{g},\mathfrak{g}]} \equiv 0$ (recall $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{h}$)

Hence $\mathfrak{g}/_{[\mathfrak{g},\mathfrak{g}]}$ (abelian Lie alg) acts on W and we can find a joint eigenvector $0 \neq v \in W$ as claimed. □