Lecture 8

1. Recall · in the previous lecture we proved Lie's theorem:

Assume k is an algebraically closed field of characteristic 0. Let $\mathfrak{g}$ be a finite-dimensional, solvable Lie algebra over k. Let V be a finite-dimensional, non-zero representation of $\mathfrak{g}$. Then $\exists\ 0 \neq v \in V$ and $\lambda \in \mathfrak{g}^*$ so that $x \cdot v = \lambda(x) v \quad \forall\ x \in \mathfrak{g}$.

Corollaries. - (1) [Same assumptions on $\mathfrak{g}$ and k as above] If V is a finite-dim'l irreducible $\mathfrak{g}$-representation, then $\dim V = 1$.

(2) [k alg closed, char 0; $\mathfrak{g}$: f.d. solvable] There exists a chain of ideals $0 = I_m \subset I_{m-1} \subset \cdots \subset I_0 = \mathfrak{g}$ s.t.

$$\dim (I_j / I_{j+1}) = 1.$$

(3) [same assumptions on $\mathfrak{g}$, k and V] There exists a chain of subrepresentations $0 = V_N \subset V_{N-1} \subset \cdots \subset V_0 = V$ s.t.

$\dim (V_j / V_{j+1}) = 1$. Thus $\dim V = N$ and there is a basis $\{v_0, v_1, \ldots, v_{N-1}\}$ of V, linear forms $\lambda_0, \ldots, \lambda_{N-1} \in \mathfrak{g}^*$ s.t.

$$x \cdot v_j = \lambda_j(x) \cdot v_j \text{ mod span of } \{v_{j+1}, \ldots, v_{N-1}\}.$$

2. Engel's theorem. - (no assumptions on the base field k) . -

Let V be a finite-dim'l (non-zero) vector space, $\mathfrak{g} \subset \mathfrak{gl}(V)$ a subalgebra so that every $x \in \mathfrak{g}$ is nilpotent (operator on V). Then $\exists\ 0 \neq v \in V$ s.t. $x \cdot v = 0\ \ \forall\ x \in \mathfrak{g}$.

Proof.- By induction on the dimension of $\mathfrak{g}$ (case of 1-dimension is obvious).

(1) Let $\mathfrak{h} \subset \mathfrak{g}$ be a maximal proper subalgebra

Claim.- $\mathfrak{h}$ is of codimension 1 - ~~dimension one~~ ideal in $\mathfrak{g}$

Pf. of the claim.- For $a \in \mathfrak{h}$, consider $\mathrm{ad}_{\mathfrak{g}}(a) : \mathfrak{g} \to \mathfrak{g}$
$x \mapsto [a,x]$
$= ax - xa$ in $\mathrm{End}(V)$

Ex. $$\mathrm{ad}_{\mathfrak{g}}(a)^N \cdot b = \sum_{\ell=0}^{N} (-1)^\ell \binom{N}{\ell} a^{N-\ell} b\, a^\ell$$

As $a \in \mathfrak{h} \subset \mathfrak{gl}(V)$ is nilpotent by assumption, so is $\mathrm{ad}_{\mathfrak{g}}(a)$. By induction ($\dim \mathfrak{h} < \dim \mathfrak{g}$) and viewing $\mathfrak{g}/\mathfrak{h}$ as a repn. of $\mathfrak{h}$ via adjoint, we can find $0 \neq \bar{x} \in \mathfrak{g}/\mathfrak{h}$ s.t. $[a, \bar{x}] = 0\ \ \forall\ a \in \mathfrak{h}$

i.e. $[a,x] \in \mathfrak{h}$ for all $a \in \mathfrak{h}$. This implies $\tilde{\mathfrak{h}} = \mathfrak{h} + k \cdot x$ is a ~~subrepresentation~~ subalgebra, which, by maximality of $\mathfrak{h}$, must be $\mathfrak{g}$.

($x \in \mathfrak{g}$ a representative of $\bar{x} \in \mathfrak{g}/\mathfrak{h}$)

Hence $\dim \mathfrak{h} = \dim \mathfrak{g} - 1$ and $\mathfrak{h} \subset \mathfrak{g}$ is an ideal. □

(ii) Now we have $\underbrace{\mathfrak{h} \subset \mathfrak{g}}_{\text{codim } 1} \subset \mathfrak{gl}(V)$ and every element of $\mathfrak{h}$ is nilpotent. By induction hypothesis, the subspace

$$V_0 := \{ v \in V : a \cdot v = 0 \ \forall a \in \mathfrak{h} \} \text{ is non-zero}$$

which (Lemma 5 of Lecture 7) is a $\mathfrak{g}$-subrepresentation of V.

As $\mathfrak{g} = \mathfrak{h} \oplus k \cdot x$ and x is nilpotent, we can find a non-zero $v \in V_0$ s.t. $x \cdot v = 0$. □

3. Corollary - A finite-dim'l Lie algebra $\mathfrak{g}$ is nilpotent if and only if $\mathrm{ad}(x)$ is nilpotent $\forall\, x \in \mathfrak{g}$

Proof. ($\Rightarrow$) already proven last time.

($\Leftarrow$) By Engel's theorem, we can find a basis of $\mathfrak{g}$, say $\{v_0, \dots, v_{N-1}\}$ such that

$$\mathrm{ad}(x) \cdot v_j \in \text{Span of } \{v_{j+1}, \dots, v_{N-1}\} \quad \forall\, 0 \le j \le N-1, \ \forall\, x \in \mathfrak{g}$$

Take $\mathfrak{g}_j$ = Span of $\{v_j, \dots, v_{N-1}\}$. Then

$$\mathfrak{g}_0 = \mathfrak{g} \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_{N-1} \supset \mathfrak{g}_N = \{0\}$$

satisfies $[\mathfrak{g}, \mathfrak{g}_j] \subset \mathfrak{g}_{j+1}$. hence $\mathfrak{g}$ is nilpotent. □