Lecture 9

Reductive and semisimple Lie algebras

1. Let k be an arbitrary field and $\mathfrak{g}$ a finite-dimensional Lie algebra over k.

Proposition - Definition. - There is a unique maximal solvable ideal in $\mathfrak{g}$, called the radical of $\mathfrak{g}$, denoted by $\mathrm{rad}(\mathfrak{g})$.

Proof. - If $\mathfrak{a}_1, \mathfrak{a}_2$ are two solvable ideals in $\mathfrak{g}$, then so is $\mathfrak{a} = \mathfrak{a}_1 + \mathfrak{a}_2$ since $\mathfrak{a}_1 \subset \mathfrak{a}$ and $\mathfrak{a}/\mathfrak{a}_1 \simeq \mathfrak{a}_2/\mathfrak{a}_1 \cap \mathfrak{a}_2$ are solvable. □

We say $\mathfrak{g}$ is semisimple if $\mathrm{rad}(\mathfrak{g}) = 0$. We say $\mathfrak{g}$ is simple if $\mathfrak{g} \neq \{0\}$ and the only ideals in $\mathfrak{g}$ are $\{0\}$ and $\mathfrak{g}$.
or 1-dim'l
We say $\mathfrak{g}$ is reductive if $\mathrm{rad}(\mathfrak{g}) = \mathcal{Z}(\mathfrak{g})$

Remark. Since $I = \mathrm{rad}(\mathfrak{g})$ is solvable, it admits a chain of ideals

$$I_0 = I \supset I_1 \supset \cdots \supset I_m \supsetneq \{0\} = I_{m+1} \qquad (m \geq 0)$$

s.t. I_j / I_{j+1} is abelian $\forall\ 0 \leq j \leq m$. Hence I_m is an abelian ideal.

Thus, we conclude that $\mathfrak{g}$ is semisimple if and only if the only abelian ideal in $\mathfrak{g}$ is $\{0\}$.

Note: Abelian $\Rightarrow$ Nilpotent $\Rightarrow$ Solvable.

Simple $\Rightarrow$ Semisimple $\Rightarrow$ Reductive.

Reductive and Solvable $\Rightarrow$ Abelian

Semisimple and Solvable $\Rightarrow$ $\{0\}$

2. Some more remarks - (a) Since simple Lie algebras have no non-trivial, proper ideals, we get:

(i) $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ (Pf.: either $[\mathfrak{g}, \mathfrak{g}] = 0$ or $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. In the first case $\mathfrak{g}$ is abelian, hence 1-dim'l contradicting its simplicity.)

(ii) Every representation of $\mathfrak{g}$ is either trivial (i.e. $\mathfrak{g}$ acts as zero) or faithful (i.e. $\mathfrak{g} \to \mathfrak{gl}(V)$ is injective).

(Pf. $\operatorname{Ker}(\mathfrak{g} \to \mathfrak{gl}(V))$ is an ideal, so it is either $\{0\}$ or $\mathfrak{g}$. $\square$)

(b) If $\mathfrak{g}$ is a semisimple Lie algebra, then $\mathfrak{g}$ has no non-trivial abelian ideals - hence $Z(\mathfrak{g}) = \{0\}$ i.e. $\operatorname{ad}: \mathfrak{g} \to \operatorname{Der}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ is injective.

(c) (Ex.) $\operatorname{Rad}(\mathfrak{g}_1 \oplus \mathfrak{g}_2) = \operatorname{Rad}(\mathfrak{g}_1) \oplus \operatorname{Rad}(\mathfrak{g}_2)$ implies that the direct sum of semisimple Lie algebras is again semisimple.

3. Examples. -

(a) $\mathfrak{sl}_2(\mathbb{C})$ = Lie algebra of 2×2 matrices of trace 0.

A basis of $\mathfrak{sl}_2(\mathbb{C})$: $h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

Lie bracket : $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$.

Claim: $\mathfrak{sl}_2(\mathbb{C})$ is simple.

Proof - Let $I \subset \mathfrak{sl}_2(\mathbb{C})$ be a non-zero ideal, and let

$z = x_1 h + x_2 e + x_3 f \in I$ be a non-zero element.
(i.e. not all x_1, x_2, x_3 are 0)

Thus $[h, z] = 2x_2 e - 2x_3 f \in I$

$[h, [h, z]] = 4x_2 e + 4x_3 f \in I$. Therefore, from the invertibility of $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$, we conclude that one of h, e or f (whichever has non-zero x_j) is in I. From the commutation relations, this implies that $\{h, e, f\} \subset I$ and hence $I = \mathfrak{sl}_2(\mathbb{C})$. □

(b) $\mathfrak{gl}_2(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{C}) \oplus \underbrace{Z(\mathfrak{gl}_2(\mathbb{C}))}_{= \mathbb{C} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$

is reductive.

(c) $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$ is semisimple.

4. Proposition - [$\mathfrak{g}$ arbitrary f.d.; k alg. closed of char. 0]

Let V be a finite-dimensional, irreducible representation of $\mathfrak{g}$.

Then $\mathfrak{a} = \mathrm{Rad}(\mathfrak{g})$ acts on V via a linear form $\lambda \in \mathfrak{a}^*$

(i.e. $\exists\ \lambda \in \mathfrak{a}^*$ s.t. $a \cdot v = \lambda(a) v \quad \forall\ a \in \mathfrak{a}$).

Moreover, $\lambda([\mathfrak{g}, \mathfrak{a}]) = 0 \quad \forall\ a \in [\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{a}$.

Proof. - As $\mathfrak{a}$ is solvable, by Lie's theorem, we know that there exists $\lambda \in \mathfrak{a}^*$ so that $V_\lambda := \{v \in V : a \cdot v = \lambda(a) v, \forall\ a \in \mathfrak{a}\}$ is non-zero. By Lemma 5 of Lecture 7, $V_\lambda \subset V$ is a sub-$\mathfrak{g}$-representation - hence $V_\lambda = V$ by irreducibility.

Note: $\forall\ a \in \mathfrak{a}$, Trace of a acting on V $= \lambda(a) \cdot \dim(V)$

and $\mathrm{Tr}([x,a]) = 0$, $\forall\ x \in \mathfrak{g}, a \in \mathfrak{a}$

implies that $\lambda([x,a]) = 0 \quad \forall\ x \in \mathfrak{g}, a \in \mathfrak{a}$. □