# Lecture 10

1. Recap. - Let $\mathfrak{g}$ be a Lie algebra over a field $k$.

$\mathrm{Rad}(\mathfrak{g})$ := largest solvable ideal of $\mathfrak{g}$.

$\mathfrak{g}$ is <u>reductive</u> (resp. <u>semisimple</u>) if $\mathrm{rad}(\mathfrak{g}) = Z(\mathfrak{g})$ (resp. $\mathrm{rad}(\mathfrak{g}) = \{0\}$).

We say $\mathfrak{g}$ is <u>simple</u> if $\dim(\mathfrak{g}) \geq 2$ and $\mathfrak{g}$ does not have any proper, non-zero ideals.

$$\text{Simple} \Rightarrow \text{Semisimple} \Rightarrow \text{Reductive}.$$

Last time we showed that if $V$ is an irreducible, finite-dim'l representation of $\mathfrak{g}$, and $k$ is alg. closed of char. 0, then

$\exists\ \lambda \in \mathrm{Rad}(\mathfrak{g})^*$ s.t. $a \cdot v = \lambda(a) v$ $\forall\ v \in V$, $a \in \mathrm{Rad}(\mathfrak{g})$.

Moreover $\lambda(a) = 0$ $\forall\ a \in [\mathfrak{g}, \mathrm{Rad}(\mathfrak{g})] \subset \mathrm{Rad}(\mathfrak{g})$.

2. <u>Invariant bilinear forms</u> - [$\mathfrak{g}$ arbitrary Lie alg ; $k$ arbitrary field]

Definition. - Let $\beta : \mathfrak{g} \times \mathfrak{g} \to k$ be a symmetric, bilinear form.

We say $\beta$ is an <u>invariant form</u> if

$$\beta([X,Y], Z) = \beta(X, [Y,Z]) \quad \forall\ X, Y, Z \in \mathfrak{g}.$$

Remark. - If $\dim(\mathfrak{g}) < \infty$, then the space of all symmetric bilinear forms can be identified with $\mathrm{Sym}^2(\mathfrak{g}^*)$ which is a $\mathfrak{g}$-representation. For $\beta \in \mathrm{Sym}^2(\mathfrak{g}^*)$ to be invariant, it

is necessary and sufficient that $x \cdot \beta = 0 \quad \forall x \in \mathfrak{g}$

$$\begin{array}{ccc} \mathrm{Sym}^2(\mathfrak{g}^*) & \longleftrightarrow & \text{Symmetric bilinear forms } \mathfrak{g}\times\mathfrak{g} \to k \\ \cup & & \cup \\ \mathrm{Sym}^2(\mathfrak{g}^*)^{\mathfrak{g}} & \longleftrightarrow & \text{Invariant forms} \end{array}$$

(notation- for $\mathfrak{g} \circlearrowright V$, $V^{\mathfrak{g}} := \{v \in V : x \cdot v = 0 \ \forall x \in \mathfrak{g}\}$)

Proof.- Let $\{x_1, \dots, x_n\}$ be a basis of $\mathfrak{g}$ ; $\beta : \mathfrak{g}\times\mathfrak{g} \to k$ a symmetric bilinear form. Let $\beta_{ij} = \beta(x_i, x_j)$ (so $\beta_{ij} = \beta_{ji}$ by symmetry). Viewing $\beta \in \mathfrak{g}^* \otimes \mathfrak{g}^*$, we can write it as

$$T_\beta = \sum_{1 \le i,j \le n} \beta_{ij}\, \varphi_i \otimes \varphi_j \quad \text{where} \quad \varphi_1, \dots, \varphi_n \in \mathfrak{g}^* \text{ are dual to } x_1, \dots, x_n \in \mathfrak{g} \text{ resp.}$$

Now $\mathfrak{g} \circlearrowright \mathfrak{g}$ via adjoint representation and hence on $\mathfrak{g}^*$

$$((x \cdot \xi)(Y) = -\xi([x,Y]) \ ; \ \forall x, Y \in \mathfrak{g}, \ \xi \in \mathfrak{g}^*)$$

and on $\mathfrak{g}^* \otimes \mathfrak{g}^*$. Thus $T \in \mathfrak{g}^* \otimes \mathfrak{g}^*$ is $\mathfrak{g}$-invariant if and only if $x \cdot T = 0 \iff (x \cdot T)(Y \otimes Z) = 0 \quad \forall x, Y, Z \in \mathfrak{g}$

$\forall x \in \mathfrak{g}$

i.e. $T([x,Y] \otimes Z + Y \otimes [x,Z]) = 0$

or $T([Y,x] \otimes Z) = T(Y \otimes [x,Z])$ as claimed

□

## 3. Invariant forms associated to finite-dimensional representations

Given a finite-dim'l representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$,

define $\beta_V : \mathfrak{g} \times \mathfrak{g} \to k$ by

$$\beta_V(x,y) = Tr_V(\rho(x) \circ \rho(y))$$

**Lemma.** $\beta_V$ is an invariant form on $\mathfrak{g}$.

Proof.- Given $x, y, z \in \mathfrak{g}$, $\beta_V([x,y], z) = Tr(\rho([x,y]) \circ \rho(z))$

$$= Tr((\rho(x)\rho(y) - \rho(y)\rho(x))\rho(z))$$

$$= Tr(\rho(x)\rho(y)\rho(z)) - Tr(\rho(x)\rho(z)\rho(y))$$

$$= Tr(\rho(x) \circ \rho([y,z])) = \beta_V(x, [y,z]). \quad \square$$

Examples. - (a) $\mathfrak{g} = k \cdot x$ 1-dim'l, abelian Lie algebra,

$\alpha \in k$ and $\rho_\alpha : \mathfrak{g} \to \mathfrak{gl}(k^1 = \text{1-dim'l v.s.})$ 1-dim'l repn

$$x \longmapsto \alpha$$

Then $\beta_{\rho_\alpha} : \mathfrak{g} \times \mathfrak{g} \to k$ is given by

$$(x,x) = \alpha^2.$$

(b) $\mathfrak{g}$ : n-dim'l abelian, $\lambda \in \mathfrak{g}^* \rightsquigarrow$ 1-dim'l repn

$$\rho_\lambda : \mathfrak{g} \to \mathfrak{gl}(k^1)$$
$$x \longmapsto \lambda(x)$$

Corresponding bilinear form is

$$\mathfrak{g} \times \mathfrak{g} \to k$$
$$(x,y) \mapsto \lambda(x)\lambda(y)$$

(c) $\mathfrak{g} = \mathfrak{gl}_n(k) \curvearrowright k^n$ "vector representation"

$\rightsquigarrow$ trace form $\quad Tr : \mathfrak{gl}_n(k) \times \mathfrak{gl}_n(k) \to k$

$$(x, y) \mapsto Tr(x \cdot y)$$

Claim. This form is non-degenerate

Proof. Let $e_{ij} \in \mathfrak{gl}_n(k)$ be elementary matrix, 1 at $(i,j)$ spot and 0's everywhere. Then

$$Tr(e_{ij} e_{kl}) = \delta_{jk} \delta_{il}$$

Means $\{e_{ij}\}_{1 \le i,j \le n}$ basis of $\mathfrak{gl}_n(k)$ admits a dual basis $\{e_{ji}\}_{1 \le i,j \le n}$ - hence Trace form is non-degenerate. □

Definition - The Killing form $K : \mathfrak{g} \times \mathfrak{g} \to k$ is the invariant form associated to the adjoint representation.

$$K(x,y) := \text{Trace}_{\mathfrak{g}} (ad(x) \circ ad(y))$$

4. Let $\mathfrak{g}$ be a Lie algebra and $V$ a finite-dim'l representation of $\mathfrak{g}$

Consider a composition series - chain of sub-representations

$$V = V_0 \supset V_1 \supset \dots \supset V_n = \{0\} \text{ , where}$$

$V_j / V_{j+1} = W_j$ is irreducible $\forall\ 0 \le j \le n-1$. (Ex. - such chain exists).

**Proposition** - For every $x, y \in g$, $\beta_V(x,y) = \sum_{j=0}^{n-1} \beta_{W_j}(x,y)$.

Proof - Upon choosing a basis of $V$ compatible with the given chain, i.e. $\{ v_1^{(0)}, \ldots, v_{m_0}^{(0)}; v_1^{(1)}, \ldots v_{m_1}^{(1)}; \ldots, v_1^{(n-1)}, \ldots, v_{m_{n-1}}^{(n-1)} \}$

where $\{ v_1^{(r)}, \ldots, v_{m_r}^{(r)} \}$ mod $V_{r+1}$ is a basis of $W_r$.

(thus $m_0 + m_1 + \ldots + m_{n-1} = \dim(V)$). every operator from $x \in g$ can be written as a block triangular matrix:

$$\rho : g \to gl(V) \quad ; \quad \rho_r : g \to gl(W_r) \quad 0 \leq r \leq n-1$$

$$\rho(x) = \begin{bmatrix} \rho_0(x) & & & 0's \\ & \rho_1(x) & & \\ & & \ddots & \\ * & & & \rho_{n-1}(x) \end{bmatrix} \quad \forall\, x \in g$$

$$\Rightarrow \text{Trace of } \rho(x) \cdot \rho(y) = \sum_{j=0}^{n-1} \text{Trace}\,(\rho_j(x)\rho_j(y))$$

i.e. $$\beta_V(x,y) = \sum_{j=0}^{n-1} \beta_{W_j}(x,y) \quad \text{as claimed.} \qquad \square$$

5. **Lemma.** If $g$ is nilpotent then the Killing form is zero.

Proof - By definition, we have a chain of ideals

$\mathfrak{g} = I_0 \supseteq \dots \supseteq I_m = \{0\}$ so that

$[\mathfrak{g}, I_j] \subset I_{j+1}$. Hence $\operatorname{ad}(x) \circ \operatorname{ad}(y) : I_j \to I_{j+2}$

$\forall\ x, y \in \mathfrak{g}$.

So, in a basis of $\mathfrak{g}$ compatible with this chain, the matrix of $\operatorname{ad}(x) \circ \operatorname{ad}(y)$ is strictly (upper) briangular, hence of trace zero.

$$\Rightarrow \quad K(x,y) = 0 \quad \forall\ x, y \in \mathfrak{g}. \qquad \square$$

6. Theorem.- If there is a finite-dimensional representation $V$ of $\mathfrak{g}$ so that the associated invariant form $\beta_V$ is non-deg. then $\mathfrak{g}$ is reductive. [$k$ alg closed of char. 0]

Proof- We have to prove that $\operatorname{rad}(\mathfrak{g}) = Z(\mathfrak{g})$ i.e. $\forall\ x \in \mathfrak{g}, y \in \operatorname{rad}(\mathfrak{g})$, $[x,y] = 0$. By non-degeneracy, this is equivalent to

$$\beta_V([x,y], z) = \operatorname{Tr}(\rho([x,y])\,\rho(z)) = 0 \quad \forall\ z \in \mathfrak{g}$$

where $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is the action hom.

In the set up of Prop. 4 above $\beta_V = \sum_{j=0}^{n-1} \beta_{W_j}$, $W_j$'s are irred., which implies that it is enough to show $\beta_V([x,y], z) = 0$ under the assumption that $V$ is irreducible.

If $V$ is irreducible, then $\mathrm{rad}(\mathfrak{g})$ acts via a linear form $\lambda \in \mathrm{rad}(\mathfrak{g})^*$ [i.e., $a \cdot v = \lambda(a)v \quad \forall\, a \in \mathrm{rad}(\mathfrak{g}),\ v \in V$.]

and hence $[a, x]$ acts as zero $\forall\, a \in \mathrm{rad}(\mathfrak{g}),\ x \in \mathfrak{g}$.

Thus, $\mathrm{Tr}\big(\rho([x,y])\,\rho(z)\big) = 0$ as claimed. □

Cor.- $\mathfrak{gl}_n(k)$ is reductive $\forall\, n \geq 1$ ($k$ alg. closed char 0).