Lecture 11

1. Recall that in the previous lecture, we introduced the notion of an invariant form on a Lie algebra: $\beta : \mathfrak{g} \times \mathfrak{g} \to k$ a symmetric, bilinear form is said to be <u>invariant</u> if $\beta([x,y],z) = \beta(x,[y,z])$; $\forall x,y,z \in \mathfrak{g}$.

If $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a finite-dimensional representation, then we obtain an invariant form $\beta_V : \mathfrak{g} \times \mathfrak{g} \to k$

$$\beta_V(x,y) := \mathrm{Tr}(\rho(x)\rho(y)) \; ; \; \forall x,y \in \mathfrak{g}.$$

When $\mathfrak{g}$ is finite-dim'l and $\rho = \mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, the resulting form is known as <u>the Killing form</u>, denoted by $K : \mathfrak{g} \times \mathfrak{g} \to k$,

$$K(x,y) = \mathrm{Tr}(\mathrm{ad}(x) \circ \mathrm{ad}(y)).$$

2. <u>Lemma</u>.- If $\mathfrak{g}$ is a Lie algebra, $\beta : \mathfrak{g} \times \mathfrak{g} \to k$ an invariant form, and $\mathfrak{a} \subset \mathfrak{g}$ an ideal, then

$$\mathfrak{a}^{\perp_\beta} := \{ x \in \mathfrak{g} \; : \; \beta(x,y) = 0 \;\; \forall y \in \mathfrak{a} \}$$

is again an ideal in $\mathfrak{g}$. [Or just $\mathfrak{a}^\perp$ if β is clear from the context.]

<u>Proof</u> - It is clear that $\mathfrak{a}^\perp \subset \mathfrak{g}$ is a linear subspace. Now given $x \in \mathfrak{a}^\perp$, $y \in \mathfrak{g}$ and $z \in \mathfrak{a}$:

$$\beta([x,y],z) = \beta(x,[y,z]) = 0 \text{ since } [y,z] \in \mathfrak{a}$$

Hence $[x,y] \in \mathfrak{a}^\perp$ and we are done. □

When $\mathfrak{g} = \mathfrak{a}$, $\mathfrak{g}^{\perp} = \{ x \in \mathfrak{g} : \beta(x,y) = 0 \ \forall\, y \in \mathfrak{g} \}$ is called the radical (or kernel) of β.

3. Jordan-Chevalley decomposition theorem. - Assume k is a perfect field. Let V be a finite-dimensional k-vector space and $A \in \mathrm{End}(V)$ be a linear operator. Then $\exists!$ $A_S, A_N \in \mathrm{End}(V)$ so that

(i) A_S is semisimple and A_N is nilpotent.

(ii) $[A_S, A_N] = 0$, and

(iii) $A = A_S + A_N$.

Moreover, $A_S = P(A)$ for some polynomial $P(t) \in k[t]$. In particular, $X \in \mathrm{End}(V)$ commutes with A if, and only if, it commutes with A_S and A_N.

* Recall - A field k is said to be perfect if every irreducible polynomial $f(x) \in k[x]$ has distinct roots in $\bar{k}$; or, equivalently, $\gcd(f(x), f'(x)) = 1$. Every alg. closed field, field of char. 0, and every finite field is perfect. The smallest example of an imperfect field is $\mathbb{F}_p(t)$, $f(x) = x^p - t \in \mathbb{F}_p(t)[x]$ is irreducible, but has repeated roots $f(x) = (x - t^{1/p})^p$. *

Proof (assuming k is algebraically closed). -

Let $\mu_A(x) \in k[x]$ be the minimal polynomial of A. Write

$$\mu_A(x) = \prod_{i=1}^{\ell} (x - \lambda_i)^{n_i} \quad ; \quad \lambda_1, \dots, \lambda_\ell \in k \text{ (eigenvalues of } A)$$

$$n_1, \dots, n_\ell \in \mathbb{Z}_{\geq 1} \, ; \, n_1 + \dots + n_\ell \leq n$$

$$n = \dim V$$

Let $p_j(x) = \dfrac{\mu_A(x)}{(x-\lambda_j)^{n_j}} = \prod_{\substack{i=1,\dots,\ell \\ i \neq j}} (x-\lambda_i)^{n_i}$, for each $1 \le j \le \ell$.

Note: $(p_1(x), \dots, p_\ell(x))$ are coprime - i.e., generate the unit ideal in $k[x]$, hence, we can find $q_1(x), \dots, q_\ell(x) \in k[x]$ so that

$$\sum_{i=1}^{\ell} q_i(x)\, p_i(x) = 1 .$$

Now, we set $E_i := q_i(A)\, p_i(A) \in \mathrm{End}(V)$.

Claim - $\{E_1, \dots, E_\ell\}$ is a complete system of pairwise orthogonal idempotents - i.e., $E_i E_j = \begin{cases} E_i & \text{if } i=j \\ 0 & \text{o/w} \end{cases}$ and $\mathrm{Id}_V = \sum_{i=1}^{\ell} E_i$.

(Proof of the claim. - Set $x = A$ in $\sum_{i=1}^{\ell} q_i(x)\, p_i(x) = 1$, to get $\sum_{i=1}^{\ell} E_i = \mathrm{Id}_V$. If $i \neq j$, $E_i E_j = q_i(A)\, q_j(A)\, p_i(A)\, p_j(A) = 0$ since $p_i(x)\, p_j(x)$ is divisible by $\mu_A(x)$. Therefore,

$$E_i = E_i \cdot \mathrm{Id}_V = E_i \cdot \sum_{j=1}^{\ell} E_j = E_i^2 . \qquad \square)$$

Using the claim, we get a direct sum decomposition

$$V = \bigoplus_{i=1}^{\ell} \mathrm{Im}(E_i)$$

Ex. - $\mathrm{Im}(E_i) = \mathrm{Ker}\left((A-\lambda_i)^{n_i}\right) = \bigcap_{\substack{j=1,\dots,\ell \\ j \neq i}} \mathrm{Ker}(E_j)$

$$= V(\lambda_i) = \left\{ v \in V : (A-\lambda_i)^{n_i} \cdot v = 0 \right\}$$

generalized eigenspace

In conclusion, $V = \bigoplus_{i=1}^{\ell} V(\lambda_i)$, A maps $V(\lambda_i)$ to itself and the minimal polynomial of $A|_{V(\lambda_i)}$ is $(x-\lambda_i)^{n_i}$. Now, define

$$P(t) := \sum_{i=1}^{\ell} \lambda_i \, q_i(t) \, p_i(t)$$

so that $P(A) = \sum_{i=1}^{\ell} \lambda_i E_i = \sum_{i=1}^{\ell} \lambda_i \, Id_{V(\lambda_i)} =: A_S$ is a semisimple operator which commutes with A (it is a polynomial in A). Finally $A_N := A - A_S$ is nilpotent ($= A - \lambda_i \cdot Id_{V(\lambda_i)}$ on $V(\lambda_i)$). It is clear that $\forall X \in End(V)$, $[X, A] = 0 \Leftrightarrow [X, A_S] = [X, A_N] = 0$.

Uniqueness: Assume $\exists$ $A_S', A_N' \in End(V)$ satisfying (i) - (iii) of the theorem. Then, A_S', A_N', A_S, A_N all commute and

$$A_S - A_S' = A_N' - A_N.$$

The left-hand side (resp. right hand side) of this equation is a difference of two commuting semisimple (resp. nilpotent) operators, hence semisimple (resp. nilpotent). Since the only operator which is both semisimple and nilpotent is zero, we conclude that $A_S - A_S' = 0 = A_N' - A_N$.

The theorem is proved. □

4. Corollary.- Let V be a finite-dimensional, k-vector space, $A \in End(V)$ and $A = A_S + A_N$ its Jordan-Chevalley decomposition. Then $ad(A) = ad(A_S) + ad(A_N) : End(V) \to End(V)$ is the J-C decomposition of $ad(A)$. Moreover, if $R(t) \in k[t]$ is such that $ad(A_S) = R(ad(A))$, then $R(0) = 0$.

Proof.- (k-alg. closed) In a basis of eigenvectors of A_S acting on V,

$A_S = \text{diagonal}(\mu_1, \dots, \mu_n)$ [$\mu_1, \dots, \mu_n$ eigenvalues with mult.]

$\Rightarrow$ $ad(A_S)$ is diagonal with entries $\mu_i - \mu_j$, hence semisimple.**

We have already shown that ad(nilpotent) is again nilpotent. hence $ad(A_S)$, $ad(A_N)$ satisfy the conditions (i) - (iii) of Thm 3 above, and our claim follows from the uniqueness.

Finally if $ad(A) = R(ad(A_S))$, then acting on $A \in End(V)$ and using the fact that $ad(A_S) \cdot A = 0$, we get

$$0 = R(0) A \quad \text{as claimed.} \qquad \square$$

** If $\{v_1, \dots, v_n\}$ is the basis so that $A_S \cdot v_i = \mu_i v_i \ \forall\, 1 \le i \le n$, then (a) $End(V)$ has a basis $\{e_{ij} : 1 \le i, j \le n\}$ where

$$e_{ij} \cdot v_k = \delta_{jk} v_i$$

(b) $$ad(A_S) \cdot e_{ij} = (\mu_i - \mu_j) e_{ij} \,. \qquad **$$

5. Cartan's criterion of solvability. - [k alg. closed of char 0]

Let $\mathfrak{g}$ be a finite-dim'l Lie algebra and V a finite-dim'l, faithful representation of $\mathfrak{g}$.

Theorem - $\mathfrak{g}$ is solvable $\Leftrightarrow$ $\beta_V(a,b) = 0$, $\forall\, a \in \mathfrak{g}$ and $b \in [\mathfrak{g},\mathfrak{g}]$

(or. equivalently $[\mathfrak{g},\mathfrak{g}] \subset \mathrm{Ker}(\beta_V)$).

Proof. - ($\Rightarrow$) If $\mathfrak{g}$ is solvable then, by Lie's theorem, we can find a chain of subrepresentations

$$V = V_0 \supset V_1 \supset \dots \supset V_{n-1} \supset V_n = \{0\}$$

so that $\dim V_j / V_{j+1} = 1$, $\forall\ 0 \le j \le n-1$. Thus, in a basis of V compatible with this chain, element of $\mathfrak{g}$ act as triangular matrices, hence of $[\mathfrak{g},\mathfrak{g}]$ act as strictly triangular matrices.

$\Rightarrow$ $\forall\, a \in \mathfrak{g}$, $b \in [\mathfrak{g},\mathfrak{g}]$, ab is strictly triangular matrix

$$\Rightarrow \beta_V(a,b) = \mathrm{Tr}_V(ab) = 0 \quad \text{as claimed.}$$

($\Leftarrow$) We will show that every $x \in [\mathfrak{g},\mathfrak{g}]$ acts nilpotently on V. Using Engel's theorem, this would imply that $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{gl}(V)$ is nilpotent, hence $\mathfrak{g}$ is solvable:

$$[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{g} \;;\; \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \quad \Rightarrow \mathfrak{g} \text{ is solvable.}$$

nilpotent, hence solvable — abelian, hence solvable

For this part, we view $\mathfrak{g} \subset \mathfrak{gl}(V)$. Let $x \in [\mathfrak{g},\mathfrak{g}]$ and let $\lambda_1, \dots, \lambda_\ell \in k$ be eigenvalues of x acting on V.

Thus, $V = \bigoplus_{i=1}^{\ell} V(\lambda_i)$ ← generalized eigenspace decomposition

Let $x = s + n$ be the Jordan-Chevalley decomposition of x. Thus, $s = \sum_{i=1}^{\ell} \lambda_i \, \mathrm{Id}_{V(\lambda_i)}$. By Cor. 4 above, we have a polynomial $P(t) \in k[t]$, $P(0) = 0$ such that $P(\mathrm{ad}(x)) = \mathrm{ad}(s)$ on $\mathrm{End}(V)$.

We want to show all λ's are zero. Let $E = \mathbb{Q}$-span of $\{\lambda_i\}_{i=1}^{\ell} \subset k$ and $f : E \to \mathbb{Q}$ a $\mathbb{Q}$-linear form.

Define $t_f \in \mathrm{End}(V)$ by $t_f = \sum_{i=1}^{\ell} f(\lambda_i) \cdot \mathrm{Id}_{V(\lambda_i)}$. Thus,

$\mathrm{ad}(t_f) \circlearrowright \mathrm{End}(V)$ with eigenvalues $\{f(\lambda_i) - f(\lambda_j) = f(\lambda_i - \lambda_j)\}_{1 \le i,j \le \ell}$

By general interpolation result, we can find $Q(t) \in k[t]$ so that

$$Q(\lambda_i - \lambda_j) = f(\lambda_i - \lambda_j) \quad \forall\ 1 \le i, j \le \ell$$

(note: $Q(0) = 0$).

Therefore, $\mathrm{ad}(t_f) = Q(\mathrm{ad}(s)) = Q(P(\mathrm{ad}(x)))$

$= R(\mathrm{ad}(x))$ ($R(t) = Q(P(t))$)

is a polynomial in $\mathrm{ad}(x)$ without constant term. Hence

for every $a \in \mathfrak{g}$, $\mathrm{ad}(t_f) \cdot a = R(\mathrm{ad}(x)) \cdot a \in [\mathfrak{g},\mathfrak{g}]$.

As $x \in [g,g]$, we can write $x = \sum_{a=1}^{m} [y_a, z_a]$.

By assumption $[g,g] \subset \mathrm{Ker}(\beta_V)$ applied to

$\mathrm{ad}(t_f) \cdot y_a \in [g,g]$ (as noted at the end of the previous page).

we get $\sum_{a=1}^{m} \mathrm{Tr}_V\left([t_f, y_a] z_a \right) = 0$. By the invariance of β_V,

the left-hand side of this equality is equal to

$$\sum_{a=1}^{m} \mathrm{Tr}_V\left(t_f \cdot [y_a, z_a] \right) = \mathrm{Tr}_V(t_f \cdot x)$$

$$= \sum_{i=1}^{\ell} f(\lambda_i)\, \lambda_i \dim V(\lambda_i)$$

since both t_f and x respect the direct sum decomposition $V = \bigoplus_{i=1}^{\ell} V(\lambda_i)$

and $V(\lambda_i)$ block of $t_f = f(\lambda_i) \cdot Id$, $x = \lambda_i \, Id$ + strict triangular.

$$\Rightarrow \quad \sum_{i=1}^{\ell} \dim V(\lambda_i)\, f(\lambda_i)\, \lambda_i = 0$$

Apply f to this equation to get $\sum_{i=1}^{\ell} \dim V(\lambda_i) \cdot f(\lambda_i)^2 = 0$

So $f(\lambda_i) = 0 \ \forall 1 \le i \le \ell$, i.e. $f = 0$. We have shown $f \in \mathrm{Hom}_{\mathbb{Q}}(E, \mathbb{Q})$

implies $f = 0$, hence $E = \{0\}$ and we are done. □