Lecture 12

1. Recall that we proved the following result in the previous lecture:

Theorem (Cartan) - [$k = \bar{k}$; $\mathrm{char}(k) = 0$] If $\mathfrak{g}$ is a Lie algebra and V a faithful, finite-dimensional representation of $\mathfrak{g}$, then

$$\mathfrak{g} \text{ is } \neq \text{ solvable} \iff \beta_V(a,b) = 0, \ \forall\, a \in \mathfrak{g}, \ b \in [\mathfrak{g},\mathfrak{g}].$$

Cor. Assume that $\mathfrak{g}$ is a finite-dim'l Lie algebra. Then

$$\mathfrak{g} \text{ is solvable} \iff K(a,b) = 0 \quad \forall\, a \in \mathfrak{g}, \ b \in [\mathfrak{g},\mathfrak{g}]$$

(K is the Killing form.)

Proof.- Use the theorem above for the faithful representation, via adjoint, of $\mathfrak{g}/Z(\mathfrak{g})$ on $\mathfrak{g}$. Recall $\mathfrak{g}/Z(\mathfrak{g})$ solvable implies $\mathfrak{g}$ is solvable. □

2. Cartan's criterion for (semi) simplicity - Let $\mathfrak{g}$ be a finite-dim'l Lie algebra over k (alg. closed of char 0). Then $\mathfrak{g}$ is semisimple if, and only if Killing form is non-degenerate.

Proof.- ($\Rightarrow$) Let $\mathfrak{a}$ = Kernel of the Killing form

$$= \{ x \in \mathfrak{g} : \mathrm{Tr}(\mathrm{ad}(x)\,\mathrm{ad}(y)) = 0 \ \forall\, y \in \mathfrak{g} \}.$$

We know $\mathfrak{a} \subset \mathfrak{g}$ is an ideal. Thus, if we pick a basis

of $\mathfrak{a}$ and complete it to a basis of $\mathfrak{g}$, the matrix of $\mathrm{ad}(x)$, for any $x \in \mathfrak{a}$ has the following form

$$\left[\begin{array}{c|c} \mathrm{ad}(x)|_{\mathfrak{a}} & \begin{matrix} * & * \\ & \end{matrix} \\ \hline \begin{matrix} 0 \end{matrix} & \begin{matrix} 0 \end{matrix} \end{array}\right]$$

(arbitrary entries)

since $\mathrm{ad}(x) : \mathfrak{g} \to \mathfrak{a}$.

basis of $\mathfrak{a}$ | completed to a basis of $\mathfrak{g}$

$\Rightarrow$ Killing form of $\mathfrak{a}$ = Killing form of $\mathfrak{g}$, hence

$$K_{\mathfrak{a}}(a,b) = 0 \quad \forall\, a, b \in \mathfrak{a}.$$

By the result obtained above in §1, $\mathfrak{a}$ is solvable. Since $\mathfrak{g}$ is semisimple, $\mathfrak{a} \subset \mathrm{rad}(\mathfrak{g}) = \{0\}$ implies $\mathfrak{a} = \{0\}$.

($\Leftarrow$) If K is non-degenerate then $\mathfrak{g}$ is reductive (Thm. 6 of Lecture 10). Moreover

$$Z(\mathfrak{g}) = \mathrm{rad}(\mathfrak{g}) \subset \text{Kernel of } K = \{0\}$$

implies $\mathfrak{g}$ is semisimple $\square$

e.g. $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. In the basis $\{f, h, e\}$ the adjoint representation takes the following form:

$$\mathrm{ad}(h) = \begin{matrix} f \\ h \\ e \end{matrix}\begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \mathrm{ad}(e) = \begin{matrix} f \\ h \\ e \end{matrix}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix}$$

$$\mathrm{ad}(f) = \begin{matrix} f \\ h \\ e \end{matrix}\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$ Hence the Killing form is given by the following matrix

$$K = \begin{matrix} f \\ h \\ e \end{matrix}\begin{bmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{bmatrix}$$ which is non-degenerate.

3. Application of Cartan's criterion.- [same assumptions on k as before]

Let $\mathfrak{g}$ be a finite-dimensional, semisimple Lie algebra over k. Let $\mathfrak{a} \subset \mathfrak{g}$ be an ideal and $\mathfrak{a}^\perp$ = orthogonal subspace with respect to the Killing form.

$$\mathfrak{a}^\perp = \{ x \in \mathfrak{g} : K(a,x) = 0 \ \forall\, a \in \mathfrak{a} \}$$

Then $\mathfrak{g} \simeq \mathfrak{a} \oplus \mathfrak{a}^\perp$ as Lie algebras.

Proof.- We know (Lemma 2 of Lecture 11) that $\mathfrak{a}^\perp \subset \mathfrak{g}$ is an ideal. If $x \in \mathfrak{a}$ and $y \in \mathfrak{a}^\perp$, then for all $z \in \mathfrak{g}$, we have

$K([x,y],z) = K(x,[y,z]) = 0$ since $x \in \mathfrak{a}$ & $[y,z] \in \mathfrak{a}^\perp$.

By non-degeneracy of K, $[x,y] = 0$. Moreover, if $I = \mathfrak{a} \cap \mathfrak{a}^\perp$ then I is an ideal in $\mathfrak{g}$ and $K_I(a,b) = K_{\mathfrak{g}}(a,b) = 0$, for $a,b \in I$ (see the proof of Thm 2 above)

(since $a \in \mathfrak{a}$, $b \in \mathfrak{a}^\perp$).

This implies (by Cartan's criterion of solvability) that I is solvable, hence equal to $\{0\}$, as $\mathfrak{g}$ is semisimple.

Hence $\mathfrak{g} \simeq \mathfrak{a} \oplus \mathfrak{a}^\perp$ as claimed. □

Remark.- Alternately, $\mathfrak{g}$ acting on itself via adjoint representation V is semisimple if is <u>semisimple</u>. (We say a representation V is semisimple if for every a subrepn. $U \subset V$, there is another subrepresentation $W \subset V$ such that $V \simeq U \oplus W$).

<u>Ex.</u> If V is a finite-dim'l semisimple representation then $V = \bigoplus_{i=1}^{N} V_i$ where $V_1, \ldots, V_N$ are f.d. irreducible representations (not necessarily distinct)