Lecture 13

1. Recall : k is a field of characteristic 0, and algebraically closed.
$\mathfrak{g}$ is a finite-dimensional Lie algebra over k.

Theorem [Cartan] .- (1) $\mathfrak{g}$ is solvable if and only if $K_{\mathfrak{g}}(\mathfrak{g}, [\mathfrak{g},\mathfrak{g}]) = 0$

(2) $\mathfrak{g}$ is semisimple if and only if $K_{\mathfrak{g}}$ is non-degenerate.
($K_{\mathfrak{g}}$ is the killing form on $\mathfrak{g}$).

We will now discuss four applications of this result.

2. **Application 1.** - Semisimplicity of the adjoint representation

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $\mathfrak{a} \subset \mathfrak{g}$ an ideal so that for every solvable ideal $I \subset \mathfrak{g}$, we have $\mathfrak{a} \cap I = \{0\}$.

Then $\mathfrak{g} \simeq \mathfrak{a} \oplus \mathfrak{a}^{\perp}$ ($\mathfrak{a}^{\perp}$ = orthogonal to $\mathfrak{a}$ relative to the Killing form $K_{\mathfrak{g}}$ on $\mathfrak{g}$).

Proof.- Since $\mathfrak{a}, \mathfrak{a}^{\perp} \subset \mathfrak{g}$ are ideals, so is $J = \mathfrak{a} \cap \mathfrak{a}^{\perp}$.

But for $a, b \in J$, $K_J(a,b) = K_{\mathfrak{g}}(a,b) = 0$ since $a \in \mathfrak{a}$, $b \in \mathfrak{a}^{\perp}$.
(Page 2 of Lecture 12 : $K_J = K_{\mathfrak{g}}|_{J \times J}$)

Hence J is solvable, so $J = \{0\}$ by our hypothesis.

$\Rightarrow$ $\mathfrak{a} \cap \mathfrak{a}^{\perp} = \{0\}$ and $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^{\perp}$ as vector spaces.

It remains to show that $[a,b] = 0$ $\forall$ $a \in \mathfrak{a}$, $b \in \mathfrak{a}^{\perp}$.

This is true since $[a,b] \in \mathfrak{a} \cap \mathfrak{a}^{\perp} = \{0\}$. $\square$

Corollary. - If $\mathfrak{g}$ is semisimple, finite-dim'l Lie algebra over k, then $\mathfrak{g} \simeq \bigoplus_{j=1}^{m} \mathfrak{g}_j$ as Lie algebras, where $\mathfrak{g}_1, \ldots, \mathfrak{g}_m$ are simple Lie algebras. The set of ideals in $\mathfrak{g} = \{ \bigoplus_{j \in J} \mathfrak{g}_j : J \subset \{1, \ldots, m\} \}$.

In particular,

(a) ideals and quotients of semisimple Lie algebras are again semisimple.

(b) $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ for a semisimple $\mathfrak{g}$.

Proof - left as an easy exercise.

3. Application 2.- All derivations of semisimple Lie algebras are inner.

Let $\mathfrak{g}$ be a finite-dim'l, semisimple Lie algebra over k.

Then $ad : \mathfrak{g} \to Der(\mathfrak{g})$ is an isomorphism

Proof.- We know ad is injective since $\ker(ad) = Z(\mathfrak{g}) = \{0\}$ as $\mathfrak{g}$ is semisimple. By the result of §2 above applied to $ad(\mathfrak{g}) \subset \mathcal{D} = Der(\mathfrak{g})$, we get $Der(\mathfrak{g}) = ad(\mathfrak{g}) \oplus I$.

($ad(\mathfrak{g})$ is an ideal by a homework problem)

Let $d \in I$ and $x \in \mathfrak{g}$. Then $[d, ad(x)] = 0 \Rightarrow ad(d(x)) = 0$

$\Rightarrow d(x) = 0$ since ad is injective

Hence $d = 0$, i.e., $I = \{0\}$ and $ad(\mathfrak{g}) = Der(\mathfrak{g})$ as claimed □

4. Application 3. - (Jordan-Chevalley decomposition).

Let $\mathfrak{g}$ be a finite-dim'l semisimple Lie algebra over k.

For any $x \in \mathfrak{g}$, there are unique elements $x_S, x_N \in \mathfrak{g}$ such that

(i) $x = x_S + x_N$,

(ii) $[x_S, x_N] = 0$,

(iii) $ad(x_S)$ (resp. $ad(x_N)$) is semisimple (resp. nilpotent)

Proof.- Consider the generalized eigenspace decomposition of $\mathfrak{g}$ with respect to $ad(x): \mathfrak{g} \to \mathfrak{g}$:

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathbb{C}} \mathfrak{g}^\lambda \quad ; \quad \mathfrak{g}^\lambda = \{ z \in \mathfrak{g} : (ad(x) - \lambda)^n \cdot z = 0 \text{ for some } n \}$$

Thus, in the Jordan-Chevalley decomposition $ad(x) = S + N$,

$$S = \bigoplus_{\lambda \in \mathbb{C}} \lambda \cdot Id_{\mathfrak{g}^\lambda}.$$

Claim.- $a \in \mathfrak{g}^\lambda$ and $b \in \mathfrak{g}^\mu$ implies $[a,b] \in \mathfrak{g}^{\lambda+\mu}$.

Proof.- Ex: Show that

$$(ad(x) - \lambda - \mu)^m \cdot [a,b] = \sum_{r=0}^{m} \binom{m}{r} \times$$

$$\times [(ad(x) - \lambda)^r \cdot a, (ad(x) - \mu)^{m-r} \cdot b].$$

Thus, if $(ad(x) - \lambda)^p \cdot a = 0 = (ad(x) - \mu)^q \cdot b$, then

$$(ad(x) - \lambda - \mu)^{p+q} \cdot [a, b] = 0.$$ The claim follows. □

This claim implies that $S([a,b]) = [S(a), b] + [a, S(b)]$, i.e., $S \in gl(g)$ is a derivation. Since all derivations are inner (§3 above) and ad is injective, there is a unique $x_S \in g$ so that $S = ad(x_S)$, and hence, for $x_N = x - x_S$, we have $N = ad(x_N)$. Finally

$$ad([x_S, x_N]) = [ad(x_S), ad(x_N)]$$
$$= [S, N] = 0$$

$\Rightarrow [x_S, x_N] = 0$ by injectivity of ad. □