Lecture 14

1. Recall : we were discussing a few applications of Cartan's criteria

Let $\mathfrak{g}$ be a finite-dim'l, semisimple Lie algebra over an algebraically closed field of characteristic zero.

(i) $\mathfrak{g} \simeq \bigoplus_{j=1}^{\ell} \mathfrak{g}_j$ as Lie algebras, where $\mathfrak{g}_1, \dots, \mathfrak{g}_\ell$ are **simple** Lie algebras. Thus,

$$\text{Set of ideals in } \mathfrak{g} = \left\{ \bigoplus_{j \in J} \mathfrak{g}_j \ : \ J \subset \{1, \dots, \ell\} \right\}$$

In particular, ideals and quotients of $\mathfrak{g}$ are semisimple, and $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

Note: $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ implies that $\mathfrak{g}$ has only one 1-dim'l (trivial) representation $\mathfrak{g} \to \mathfrak{gl}_1(k) = k$.
$x \mapsto 0 \quad \forall\, x \in \mathfrak{g}$

(ii) The Lie algebra homomorphism $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ is an isomorphism.

(iii) $\forall\, x \in \mathfrak{g}$, $\exists!\ x_S, x_N \in \mathfrak{g}$ s.t. $\begin{cases} x = x_S + x_N \\ [x_S, x_N] = 0 \\ \mathrm{ad}(x_S) \text{ is semisimple.} \\ \mathrm{ad}(x_N) \text{ is nilpotent.} \end{cases}$

(Jordan-Chevalley decomposition)

The final application is towards Casimir tensor and element:

2. Casimir tensor. Let $\mathfrak{g}$ be a finite-dim'l Lie algebra over k, and $\beta : \mathfrak{g} \times \mathfrak{g} \to k$ an invariant, non-degenerate form.

Thus we have a linear isomorphism

$$\psi_\beta : \mathfrak{g} \longrightarrow \mathfrak{g}^*$$
$$x \longmapsto \psi_\beta(x) = \beta(x, -)$$

Lemma. - ψ_β is an isomorphism of $\mathfrak{g}$-representations.

Proof - Let $x \in \mathfrak{g}$ and adjoint action of $\mathfrak{g}$ on itself:

$\forall\, y, z \in \mathfrak{g}$; $\psi_\beta(x \cdot y) : z \longmapsto \beta(x \cdot y, z)$

$$= \beta([x,y], z) = -\beta(y, [x,z])$$
$$= \psi_\beta(y)(-[x,z])$$
$$= (x \cdot \psi_\beta(y))(z)$$

i.e. $\psi_\beta(x \cdot y) = x \cdot \psi_\beta(y)$. The lemma is proved. □

Now consider the composition of the following $\mathfrak{g}$-intertwiners

$$\mathfrak{g} \otimes \mathfrak{g} \xrightarrow{\psi_\beta \otimes Id_{\mathfrak{g}}} \mathfrak{g}^* \otimes \mathfrak{g} \xrightarrow{A} End_{k\text{-v.s.}}(\mathfrak{g})$$

(recall: $A(\xi \otimes x) : y \longmapsto \xi(y)x$)

$\forall\, x, y \in \mathfrak{g}$; $\xi \in \mathfrak{g}^*$.

Definition - $\Omega_\beta \in \mathfrak{g} \otimes \mathfrak{g}$ is the unique element so that

$$A \circ (\psi_\beta \otimes id_{\mathfrak{g}}) : \Omega_\beta \mapsto Id_{\mathfrak{g}} \in End_{\mathfrak{g}}(\mathfrak{g}) \subset End_{k\text{-v.s.}}(\mathfrak{g})$$

Note: $\mathfrak{g}$ acts on $End_{k\text{-v.s.}}(\mathfrak{g})$ via

$$(x \cdot f)(y) = [x, f(y)] - f([x,y]) \; ; \quad \begin{matrix} x, y \in \mathfrak{g} \\ f : \mathfrak{g} \to \mathfrak{g} \\ \text{linear map.} \end{matrix}$$

Thus $x \cdot Id_{\mathfrak{g}} = 0 \quad \forall \; x \in \mathfrak{g}$. This implies the following property of Ω_β (since $\psi_\beta \otimes Id_{\mathfrak{g}}$ and A are $\mathfrak{g}$-intertwiners)

Prop. - $[x \otimes 1 + 1 \otimes x, \Omega_\beta] = 0 \quad \forall \; x \in \mathfrak{g}$.

Direct proof. Let $\{a_i\}_{i=1,\dots,n}$ be a basis of $\mathfrak{g}$ and $\{a^j\}_{j=1,\dots,n}$ the dual basis relative to the form β. Thus,

$$\beta(a_i, a^j) = \delta_{ij} \; , \quad 1 \le i, j \le n.$$

Ex: $A \circ (\psi_\beta \otimes id) : \sum_{i=1}^{n} a_i \otimes a^i \longmapsto Id_{\mathfrak{g}}$.

For $x \in \mathfrak{g}$, define the matrix coefficients of $ad(x) \circlearrowright \mathfrak{g}$ in these two bases as:

$$[x, a_i] = \sum_{j=1}^{n} c_{ij}(x)\, a_j$$

$$[x, a^i] = \sum_{j=1}^{n} d_{ij}(x)\, a^j$$

Thus $c_{ij}(x) = \beta([x, a_i], a^j) = -\beta([a_i, x], a^j)$

$$= -\beta(a_i, [x, a^j]) \quad \text{by invariance of } \beta.$$

$$= -\beta\left(a_i, \sum_{k=1}^{n} d_{jk}(x)\, a^k\right) = -d_{ji}(x)$$

$$\text{i.e., } \quad c_{ij}(x) = -d_{ji}(x), \quad \forall\ x \in \mathfrak{g}. \qquad - \quad (*)$$

$$\text{Now} \quad \left[x \otimes 1 + 1 \otimes x,\ \Omega_\beta\right] = \sum_{i=1}^{n} \left([x; a_i] \otimes a^i + a_i \otimes [x, a^i]\right)$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} c_{ij}(x) a_j\right) \otimes a^i + a_i \otimes \left(\sum_{j=1}^{n} d_{ij}(x)\, a^j\right)$$

Coefficient of $a_s \otimes a^t$ is: $\quad c_{ts}(x) + d_{st}(x) = 0$ by $(*)$.

□

When $\mathfrak{g}$ is semisimple and $\beta = K_{\mathfrak{g}}$ the Killing form on $\mathfrak{g}$, Ω_K is called <u>the Casimir tensor of $\mathfrak{g}$</u>.

<u>3. Casimir element</u> - Again let $\mathfrak{g}$ be a finite-dim'l Lie algebra together with a <u>non-degenerate invariant form</u> $\beta : \mathfrak{g} \times \mathfrak{g} \to k$. As above, let $\{a_i\}_{i=1,\dots,n}$ and $\{a^i\}_{i=1,\dots,n}$ be dual bases relative to β. Define

$$C_\beta := \sum_{i=1}^{n} a_i a^i \in U(\mathfrak{g})$$

Note: Under the linear map ($\mathfrak{g} \xrightarrow{i} \mathcal{U}(\mathfrak{g})$ is the canonical embedding)

$$\begin{array}{ccc} \mathfrak{g} \otimes \mathfrak{g} & \xrightarrow{i \otimes i} & \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) \\ & \searrow & \downarrow \text{multiplication} \\ & & \mathcal{U}(\mathfrak{g}) \end{array}$$

$$\Omega_\beta \longmapsto C_\beta .$$

Proposition. - $C_\beta \in \mathcal{Z}(\mathcal{U}(\mathfrak{g}))$.

Proof. - Since $\mathcal{U}(\mathfrak{g})$ is generated by $\mathfrak{g} \subset \mathcal{U}(\mathfrak{g})$, it suffices to prove that $[x, C_\beta] = 0 \quad \forall\, x \in \mathfrak{g}$.

Thus, $$[x, C_\beta] = \sum_{i=1}^{n} [x, a_i]\, a^i + a_i\, [x, a^i]$$

$$= \sum_{1 \le s,t \le n} \left(C_{ts}(x) + d_{st}(x) \right) a_s\, a^t \qquad \left[\begin{array}{l} \text{see the proof of} \\ \text{Prop - pages 3, 4} \end{array} \right]$$

$$= 0 .$$

□

Again when $\mathfrak{g}$ is semisimple and $\beta = K_{\mathfrak{g}}$, we the corresponding C_K is called the Casimir element.

4. Eigenvalues of the Casimir element - Let $\mathfrak{g}$ be a finite-dim'l, semisimple Lie algebra.

Lemma.- Let V be a finite-dim'l, faithful, irreducible representation of $\mathfrak{g}$, and $\beta_V : \mathfrak{g} \times \mathfrak{g} \to k$ the associated invariant form. Then β_V is non-degenerate and C_{β_V} acts as $\frac{\dim(\mathfrak{g})}{\dim(V)} \cdot Id_V$ on V.

Proof.- Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be the action homomorphism. As ρ is injective, we view $\mathfrak{g} \subset \mathfrak{gl}(V)$ Lie subalgebra

If $\mathfrak{or} \subset \mathfrak{g}$ is the Kernel of β_V

$$\mathfrak{or} = \{ x \in \mathfrak{g} : Tr_V(\rho(x)\rho(y)) = 0 \quad \forall y \in \mathfrak{g} \}.$$

then $\mathfrak{or}$ is an ideal of $\mathfrak{g}$, acting faithfully on V such that $\beta_V(\mathfrak{or}, [\mathfrak{or}, \mathfrak{or}]) = 0$. Hence $\mathfrak{or}$ is solvable (by Cartan's criterion) and therefore $\mathfrak{or} = \{0\}$ as $\mathfrak{g}$ is semisimple.

Now let $\{a_i\}_{i=1,\dots,n=\dim(\mathfrak{g})}$ and $\{a^i\}$ be dual bases relative to β_V. $C = C_{\beta_V} = \sum_{i=1}^{n} a_i a^i \in Z(U(\mathfrak{g}))$ commutes with the $\mathfrak{g}$-action on V (by Prop. 3 above).

As V is irreducible, by Schur's lemma, $C = \lambda \cdot Id_V$ for some $\lambda \in k$. Thus $Tr(C) = \lambda \cdot \dim(V)$.

But $Tr(C) = \sum_{i=1}^{n} \beta_V(a_i, a^i) = n$. Hence, $\lambda = \frac{n}{\dim(V)}$. □