Lecture 15

1. Recall - we are studying properties of finite-dimensional, semisimple Lie algebras. Our next big result is Weyl's complete reducibility theorem.

Theorem. - Let $\mathfrak{g}$ be a finite-dim'l semisimple Lie algebra over k. Let V be a finite-dim'l $\mathfrak{g}$-representation. Then for every subrepresentation $U \subset V$, there exists a subrepresentation $W \subset V$ so that $U \oplus W \xrightarrow{\sim} V$ as $\mathfrak{g}$-reprs.

In other words, every finite-dim'l representation of $\mathfrak{g}$ is semisimple, hence isomorphic to a direct sum of irreducible $\mathfrak{g}$-representations.

Equivalent formulation: $\mathrm{Rep}_{fd}(\mathfrak{g})$ is a semisimple category

Or, every short-exact sequence of $\mathfrak{g}$-intertwiners

$$0 \to U \xrightarrow{i} V \xrightarrow{p} W \to 0$$

splits - ie. $\exists$ a $\mathfrak{g}$-intertwiner (section) $s : W \to V$ so that $p \circ s = \mathrm{Id}_W$.

2. First Lie algebra cohomology space - Let $\mathfrak{g}$ be an arbitrary Lie algebra and M be a representation of $\mathfrak{g}$.

Define $C^1(\mathfrak{g}; M) = \mathrm{Hom}_{k\text{-v.s.}}(\mathfrak{g}, M)$

$C^0(\mathfrak{g}; M) = M$; $C^2(\mathfrak{g}; M) = \mathrm{Hom}_{k\text{-v.s.}}(\Lambda^2 \mathfrak{g}, M)$

$$0 \to C^0(\mathfrak{g};M) = M \xrightarrow{d^0} C^1(\mathfrak{g};M) \xrightarrow{d^1} C^2(\mathfrak{g};M)$$
$$\| \qquad\qquad\qquad\qquad \|$$
$$\underset{k\text{-v.s.}}{\mathrm{Hom}}(\mathfrak{g};M) \qquad \underset{k\text{-v.s.}}{\mathrm{Hom}}(\Lambda^2\mathfrak{g},M)$$

defined as:

$\forall\, m \in M$, $(d^0(m)) : x \longmapsto x \cdot m \quad \forall\, x \in \mathfrak{g}$.

$\forall\, f : \mathfrak{g} \to M$ linear, $(d^1 f) : (x,y) \longmapsto x \cdot f(y) - y \cdot f(x) - f([x,y])$

$$Z^1(\mathfrak{g};M) = \{ f \in C^1(\mathfrak{g};M) : d^1 f = 0, \text{ i.e., } f([x,y]) = x \cdot f(y) - y \cdot f(x) \ \forall\, x,y \in \mathfrak{g} \}$$

1-cocycles

$$B^1(\mathfrak{g};M) = \{ f \in C^1(\mathfrak{g};M) : f = d^0 m \text{ for some } m \in M$$
$$\text{i.e. } \exists\, m \in M : f(x) = x \cdot m \ \forall\, x \in \mathfrak{g} \}$$

Ex: $d^1 \circ d^0 = 0$, i.e. $B^1(\mathfrak{g};M) \subset Z^1(\mathfrak{g};M)$.

The first Lie algebra cohomology space is defined as the quotient

$$H^1(\mathfrak{g};M) := Z^1(\mathfrak{g};M) \Big/ B^1(\mathfrak{g};M)$$

3. Motivation and relation with extensions [$\mathfrak{g}$: arbitrary].

Let V_1 and V_2 be two finite-dim'l $\mathfrak{g}$-representations. An <u>extension of V_2 by V_1</u> is a short exact sequence of $\mathfrak{g}$-intertwiners

$$0 \to V_1 \to V \to V_2 \to 0$$

Note that $V \simeq V_1 \oplus V_2$ as vector spaces, and if $\rho_i : \mathfrak{g} \to \mathfrak{gl}(V_i)$ are the action hom's, then $\mathfrak{g}$-action on V is of the form :

$$\rho(x) = \left[\begin{array}{c|c} \rho_1(x) & \rho_{12}(x) \\ \hline 0 & \rho_2(x) \end{array}\right], \text{ where } \rho_{12}(x) : V_2 \to V_1$$

<u>Proposition (a)</u>- For an arbitrary linear map ~~$\psi : V_2 \to V_1$~~ to define a Lie alg. hom. (i.e. a representation) $\psi : \mathfrak{g} \to \underset{\text{k-v.s.}}{\mathrm{Hom}}(V_2, V_1)$

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V_1 \oplus V_2)$$

$$\rho(x) = \left[\begin{array}{c|c} \rho_1(x) & \psi(x) \\ \hline 0 & \rho_2(x) \end{array}\right]$$

it is necessary and sufficient that $\psi \in Z^1(\mathfrak{g}\,;\, \underset{\text{k-v.s.}}{\mathrm{Hom}}(V_2, V_1))$.

<u>Proof</u> For $x, y \in \mathfrak{g}$, we begin by computing $[\rho(x), \rho(y)]$

$$[\rho(x), \rho(y)] = \left[\begin{array}{c|c} [\rho_1(x), \rho_1(y)] & \begin{array}{l}\rho_1(x)\psi(y) - \psi(y)\rho_2(x) \\ + \psi(x)\rho_2(y) - \rho_1(y)\psi(x)\end{array} \\ \hline 0 & [\rho_2(x), \rho_2(y)] \end{array}\right]$$

Thus, ρ defines a representation if and only if

$$\psi([x,y]) = \rho_1(x)\psi(y) - \psi(y)\rho_2(x) - (\rho_1(y)\psi(x) - \psi(x)\rho_2(y))$$

Recall that $\mathfrak{g}$ acts on $\underset{\text{K-v.s.}}{\mathrm{Hom}}(V_2, V_1)$ via

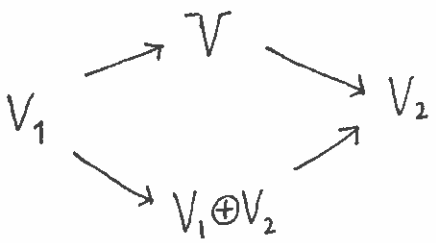
for $x \in \mathfrak{g}$; $f \in \underset{\text{K-v.s.}}{\mathrm{Hom}}(V_2, V_1)$; $(x \cdot f)(v_2) = \rho_1(x) \cdot f(v_2) - f(\rho_2(x) \cdot v_2)$

Thus, the equation above can be rewritten as

$$x \cdot \psi(y) - y \cdot \psi(x) - \psi([x,y]) = 0$$

i.e. $d^1\psi = 0$ as claimed. □

Again let $0 \to V_1 \to V \to V_2 \to 0$ be an extension corresponding to some $\psi \in Z^1(\mathfrak{g}; \mathrm{Hom}(V_2, V_1))$. We say this extension is trivial if there is an isomorphism of $\mathfrak{g}$-repns. $V \simeq V_1 \oplus V_2$ so that



commutes. Thus such an isomorphism is given by $\left[\begin{array}{c|c} \mathrm{Id} & f \\ \hline 0 & \mathrm{Id} \end{array}\right]$ for some $f \in \underset{\text{K-v.s.}}{\mathrm{Hom}}(V_2, V_1)$.

In order to avoid confusion, let $V_1 \underset{\psi}{\oplus} V_2$ denote the representation where $\mathfrak{g}$ acts on $V_1 \oplus V_2$ by $\left[\begin{array}{c|c} \rho_1 & \psi \\ \hline 0 & \rho_2 \end{array}\right]$.

Proposition (6).- $V_1 \oplus_\psi V_2 \xrightarrow{\sim} V_1 \oplus V_2$ is an iso. of $\left[\begin{array}{c|c} Id_{V_1} & f \\ \hline 0 & Id_{V_2} \end{array}\right]$ $\mathfrak{g}$-representations if and only if $\psi \in B^1(\mathfrak{g}; \underset{k\text{-v.s.}}{Hom}(V_2, V_1))$.

Proof.- Let $x \in \mathfrak{g}$. Its action on $V_1 \oplus_\psi V_2 = \rho_\psi(x) = \begin{bmatrix} \rho_1(x) & \psi(x) \\ 0 & \rho_2(x) \end{bmatrix}$

on $V_1 \oplus V_2 = \rho_0(x) = \begin{bmatrix} \rho_1(x) & 0 \\ 0 & \rho_2(x) \end{bmatrix}$

Thus, the given vector space isomorphism is a $\mathfrak{g}$-intertwiner, if and only if, $\forall\, x \in \mathfrak{g}$:

$$\begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \rho_1(x) & \psi(x) \\ 0 & \rho_2(x) \end{bmatrix} = \begin{bmatrix} \rho_1(x) & 0 \\ 0 & \rho_2(x) \end{bmatrix} \begin{bmatrix} 1 & f \\ 0 & 1 \end{bmatrix}$$

i.e. $\psi(x) + f \circ \rho_2(x) = \rho_1(x) \circ f$

i.e. $\psi(x) = x \cdot f$ meaning $\psi = d^\circ f \in B^1(\mathfrak{g}; Hom(V_2, V_1))$ □

4. **Summarizing :** The set of isomorphism classes of extensions of V_2 by V_1 ($V_1, V_2 \in Rep_{fd.}(\mathfrak{g})$)

$$Ext^1(V_2, V_1) = H^1(\mathfrak{g}; M) \text{ where } M = \underset{k\text{-v.s.}}{Hom}(V_2, V_1)$$

Remark.- $H^\circ(\mathfrak{g}; M) = M^{\mathfrak{g}} = \{m \in M : x \cdot m = 0 \ \forall\, x \in \mathfrak{g}\}$

$= Hom_{\mathfrak{g}}(\text{Trivial}, M)$

Trivial = 1-dim'l representation $\mathfrak{g} \to \mathfrak{gl}_1(k)$, $x \mapsto 0$, $\forall\, x \in \mathfrak{g}$

Cohomology of $\mathfrak{g}$ valued in M can be viewed as "derived functors" of $M \mapsto M^{\mathfrak{g}}$ $\quad Rep_{f.d.}(\mathfrak{g}) \longrightarrow Vect_K$ - which is not necessarily exact. Meaning, we have a sequence of functors

$H^n(\mathfrak{g}; -) : Rep_{f.d.}(\mathfrak{g}) \longrightarrow Vect_K$ sisters together with

connecting homomorphisms (short-exact seq.) $\forall\ 0 \to V_1 \to V \to V_2 \to 0$, we have $H^n(\mathfrak{g}; V_2) \xrightarrow{\partial_n} H^{n+1}(\mathfrak{g}; V_1)$

s.t. (i) $H^0(\mathfrak{g}, M) = M^{\mathfrak{g}}$, $\forall\ M \in Rep_{f.d.}(\mathfrak{g})$

(ii) $\forall$ short exact sequence, $0 \to M_1 \to M \to M_2 \to 0$ in $Rep_{fd}(\mathfrak{g})$

we have a long-exact sequence

$$0 \to H^0(\mathfrak{g}; M_1) \to H^0(\mathfrak{g}; M) \to H^0(\mathfrak{g}; M_2) \xrightarrow{\partial_0} H^1(\mathfrak{g}; M_1) \to \cdots$$

(iii) Connecting homomorphisms are functorial. That is, for every commutative diagram with exact rows,

$$\begin{array}{ccccccc} 0 \to & M_1 & \longrightarrow & M & \longrightarrow & M_2 & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & N_1 & \longrightarrow & N & \to & N_2 & \to 0 \end{array}$$

the following diagram commutes:

$$\begin{array}{ccc} H^n(\mathfrak{g}; M_2) & \to & H^{n+1}(\mathfrak{g}; M_1) \\ \downarrow & & \downarrow \\ H^n(\mathfrak{g}; N_2) & \to & H^{n+1}(\mathfrak{g}; N_1) \end{array}$$

For now, we only need exactness of $H^1(\mathfrak{g}; -)$ and will prove it below.

5. Proposition.- $M \mapsto H^1(\mathfrak{g}; M)$ is a functor, and for every short exact sequence $0 \to M_1 \to M \to M_2 \to 0$ the resulting sequence

$$H^1(\mathfrak{g}; M_1) \longrightarrow H^1(\mathfrak{g}; M) \longrightarrow H^1(\mathfrak{g}; M_2).$$

In particular, $H^1(\mathfrak{g}; M_1) = H^1(\mathfrak{g}; M_2) = \{0\}$ implies

$$H^1(\mathfrak{g}; M) = \{0\}.$$

Proof.- Functoriality. Recall $H^1(\mathfrak{g}; M) = \{f : \mathfrak{g} \to M$ linear s.t. $f([x,y]) = x \cdot f(y) - y \cdot f(x)\}$ modulo $\{f : \exists m$ s.t. $f(x) = x \cdot m\}$

For a $\mathfrak{g}$-intertwiner $f : M \to N$, we get

$$C^1(\mathfrak{g}; M) = \operatorname{Hom}_{K\text{-v.s.}}(\mathfrak{g}, M) \xrightarrow{h_f} \operatorname{Hom}_{K\text{-v.s.}}(\mathfrak{g}; N) = C^1(\mathfrak{g}; N)$$

$$h_f(\xi) = f \circ \xi : \mathfrak{g} \to N.$$

Assume $\xi \in C^1(\mathfrak{g}; M)$ is s.t. $\xi([x,y]) = x \cdot \xi(y) - y \cdot \xi(x)$, $\forall x, y \in \mathfrak{g}$

then $h_f(\xi) : \mathfrak{g} \to N$ also satisfies this equation:

$$\begin{aligned} h_f(\xi)([x,y]) &= f(\xi([x,y])) \\ &= f(x \cdot \xi(y) - y \cdot \xi(x)) \\ &= x \cdot h_f(\xi)(y) - y \cdot h_f(\xi)(x) \end{aligned}$$

since f is a $\mathfrak{g}$-intertwiner.

Hence $h_f : Z^1(\mathfrak{g}; M) \longrightarrow Z^1(\mathfrak{g}; N)$

Finally if $\xi = d^\circ(m)$ for some $m \in M$, i.e.

$$\xi(x) = x \cdot m \quad \forall\, x \in \mathfrak{g},$$

then $h_f(\xi) = d^\circ(f(m))$, again because f is $\mathfrak{g}$-interlinear.

So, we get $H^1(\mathfrak{g}; M) \xrightarrow{h_f} H^1(\mathfrak{g}; N)$

$$[\xi] \longmapsto [f \circ \xi]$$

([ ] = equivalence class modulo B^1)

It is easy to verify that $f \mapsto h_f$ is linear inf,

$$h_{Id} = Id \quad \text{and} \quad h_{g \circ f} = h_g \circ h_f \text{ - i.e.}$$

$H^1(\mathfrak{g}, -) : Rep_{fd}(\mathfrak{g}) \longrightarrow Vect_k$ is a functor (additive).

Exactness. Let $0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0$ be an exact seq. of $\mathfrak{g}$-representations. Let $[\xi] \in H^1(\mathfrak{g}; M)$ be such that $[h_g(\xi)] = 0$. We have to show that $\xi = h_f(\eta)$ for some $[\eta] \in H^1(\mathfrak{g}; M_1)$.

As $[h_g(\xi)] = 0$, $\exists\, m_2 \in M_2$ s.t. $h_g(\xi) : x \mapsto x \cdot m_2$ $\forall\, x \in \mathfrak{g}$.

i.e. $g(\xi(x)) = x \cdot m_2$

Since g is surjective, we can find $m \in M$ s.t. $m_2 = g(m)$.

Thus $g(\xi(x)) = g(x \cdot m) \Rightarrow g(\xi(x) - x \cdot m) = 0$

(g is an interlwiner)

By exactness, $\forall\, x \in \mathfrak{g}$, we find a unique $\eta(x) \in M_1$ s.t.

$$\xi(x) - x\cdot m = f(\eta(x)) \quad \text{for some } \eta : \mathfrak{g} \to M_1$$

It is an easy check to see that $\eta \in Z^1(\mathfrak{g}; M_1)$ [since f is an injective $\mathfrak{g}$-intertwiner]. We have shown

$$[g\circ\xi] = 0 \quad \Rightarrow \quad \xi \in B^1(\mathfrak{g}; M) + \mathrm{Im}(h_f)$$

Clearly $h_{g\circ f} = h_0 = 0$, hence $H^1(\mathfrak{g}; M_1) \xrightarrow{h_f} H^1(\mathfrak{g}; M) \xrightarrow{h_g} H^1(\mathfrak{g}; M_2)$ is exact. □

Cor. - [$\mathfrak{g}$: finite-dim'l Lie algebra] The following are equivalent

(i) $\mathrm{Rep}_{f.d.}(\mathfrak{g})$ is semisimple

(ii) $\mathrm{Ext}^1(V_2, V_1) = \{0\}$ for all $V_1, V_2 \in \mathrm{Rep}_{fd}(\mathfrak{g})$

(iii) $H^1(\mathfrak{g}; M) = \{0\}$ for every $M \in \mathrm{Rep}_{fd}(\mathfrak{g})$

(iv) $H^1(\mathfrak{g}; M) = \{0\}$ for every <u>irreducible</u> $M \in \mathrm{Rep}_{f.d.}(\mathfrak{g})$.

(i.e. $\mathrm{Ext}^1(\mathrm{Trivial}, M) = \{0\}$ $\forall$ f.d. irred. M).