Lecture 16

1. Recall - last time we defined, for a Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}$-representation M,

$$H^1(\mathfrak{g};M) := \frac{\{f:\mathfrak{g}\xrightarrow{\text{linear}} M \mid f([x,y]) = x\cdot f(y) - y\cdot f(x),\ \forall x,y\in\mathfrak{g}\}}{\{f = d^\circ m : \mathfrak{g}\to M \mid m\in M\}}$$

$$x\mapsto x\cdot m$$

We showed that $M \longmapsto H^1(\mathfrak{g};M)$ is a functor (to the category of vector spaces), and for every short exact sequence of $\mathfrak{g}$-intertwiners

$$0\to M_1\to M\to M_2\to 0,$$ we get an exact sequence of linear maps $H^1(\mathfrak{g};M_1)\to H^1(\mathfrak{g};M)\to H^1(\mathfrak{g};M_2)$.

Moreover, every short exact sequence $0\to V_1\to V\to V_2\to 0$ of $\mathfrak{g}$-intertwiners splits (for $V_1, V_2\in \mathrm{Rep}_{f.d.}(\mathfrak{g})$)

if, and only if $H^1(\mathfrak{g};\ \mathrm{Hom}_{\text{V.S.}}(V_2,V_1)) = \{0\}$.

2. Proposition - [$\mathfrak{g}$ arbitrary] If $H^1(\mathfrak{g};M) = \{0\}$ for every irreducible $M\in \mathrm{Rep}_{f.d.}(\mathfrak{g})$, then every $V\in \mathrm{Rep}_{fd.}(\mathfrak{g})$ is semisimple.

Proof. We first show that every $M\in \mathrm{Rep}_{f.d.}(\mathfrak{g})$ has a composition series - i.e., a chain of subrepresentations

$$M_0 = M \supseteq M_1 \supseteq \dots \supseteq M_\ell = \{0\} \text{ s.t. each}$$

M_j/M_{j+1} is simple $(0\le j\le \ell-1)$.

Proof - We argue by dimension. If $\dim M = 1$, then M is irred.

Induction hypothesis: every $M \in Rep_{f.d.}(\mathfrak{g})$ with $\dim(M) \leq n$ admits a composition series.

Induction step: $M \in Rep_{f.d.}(\mathfrak{g})$, $\dim(M) = n+1$

If M is irreducible, then we are done $M_0 = M \supseteq M_1 = \{0\}$.

Otherwise choose $M_1' \subset M$ non-zero proper submodule of least dimension (among all non-zero, proper submodules).

$\dim(M/M_1') \leq n$, hence it has a composition series

$\overline{M}_0 = M/M_1' \supset \overline{M}_1 \supset \dots \supset \overline{M}_r = \{0\}$. If $\pi: M \to M/M'$ is the canonical surjection, then

$M = \pi^{-1}(\overline{M}_0) \supset \pi^{-1}(\overline{M}_1) \supset \dots \supset \pi^{-1}(\overline{M}_r) = M' \supset \{0\}$ is a composition series for M. □

Next, we show, by induction on the length of a composition series, that $H^1(\mathfrak{g}; M) = \{0\}$, for every $M \in Rep_{f.d.}(\mathfrak{g})$.

Base case. $\ell = 1$ ie M is irreducible and $H^1(\mathfrak{g}, M) = \{0\}$ by hypothesis.

Induction hypothesis: $H^1(\mathfrak{g}; M) = \{0\}$ $\forall$ M which admit a composition series of length $\leq \ell$.

Step: if M has a composition series of length $\ell+1$,

$$M = M_0 \supsetneq M_1 \cdots \supsetneq M_\ell \supsetneq M_{\ell+1} = \{0\}$$

then M/M_ℓ has a composition series of length ℓ - and by induction $H^1(\mathfrak{g}, M/M_\ell) = \{0\} \overset{\uparrow}{=} H^1(\mathfrak{g}; M_\ell)$. Hence $H^1(\mathfrak{g}; M) = \{0\}$.

↑ from assumption

Now we are done by the results recalled in §1 above. □

3. Theorem. - [Whitehead's Lemma]. Let $\mathfrak{g}$ be a finite-dim'l, semisimple Lie algebra over k (alg. closed of char 0).

Then $H^1(\mathfrak{g}; M) = \{0\}$, for every irreducible $M \in \mathrm{Rep}_{f.d.}(\mathfrak{g})$.

Hence $\boxed{\mathrm{Rep}_{f.d.}(\mathfrak{g}) \text{ is a semisimple category}}$ - Weyl's complete reducibility theorem.

Proof.- If $\dim(M) = 1$, then $f : \mathfrak{g} \to M$ satisfying $f([x,y]) = x \cdot f(y) - y \cdot f(x)$
$= 0$ (only 1-dim'l repn. of $\mathfrak{g}$ is the trivial one)
i.e. $f = 0$ on $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Hence, $H^1(\mathfrak{g}, k) = \{0\}$

Now we assume that M is non-trivial, irreducible, finite-dim'l $\mathfrak{g}$-representation. Consider the action homomorphism

$$\rho : \mathfrak{g} \to \mathrm{End}(M) \ (\text{or } \mathfrak{gl}(M))$$

Let $\mathfrak{a} = \mathrm{Ker}(\rho)$ and $\mathfrak{h} = \mathrm{Im}(\mathfrak{g}\rho) \subset \mathfrak{gl}(M)$. As $\mathfrak{g}$ is semisimple

$\mathfrak{g} \simeq \mathfrak{a} \oplus \mathfrak{a}^\perp$ as Lie alg's and $\rho : \mathfrak{a}^\perp \xrightarrow{\sim} \mathfrak{h}$ is again semisimple. ($\mathfrak{a}^\perp$ is identified with $\mathfrak{h}$ below via this iso.)

Thus M is a faithful, irreducible representation of $\mathfrak{h}$.
By Lemma 4 of Lecture 14, the associated form $\beta_M : \mathfrak{h} \times \mathfrak{h} \to k$ is non-degenerate and the corresponding Casimir element C_M acts on M as $\lambda \cdot \mathrm{Id}_M$ $\left(\lambda = \dfrac{\dim(\mathfrak{h})}{\dim(M)} \neq 0\right)$.

Let $\{a_i\}_{i=1,\dots,r}$ be a basis of $\mathfrak{h}$ and $\{a^i\}_{i=1,\dots,r}$ the dual basis with respect to non-degenerate β_M.

Now, let $f : \mathfrak{g} \to M$ be a linear map satisfying 1-cocycle equation

$$f([x,y]) = x \cdot f(y) - y \cdot f(x), \quad \forall\, x, y \in \mathfrak{g}.$$

Define $m := \frac{1}{\lambda} \sum_{i=1}^{r} a_i \cdot f(a^i) \in M$.

Claim.- $x \cdot m = f(x), \quad \forall\, x \in \mathfrak{g}$.

Pf of the claim. $\lambda (x \cdot m) = \sum_{i=1}^{r} x \cdot a_i \cdot f(a^i)$

$$= \sum_{i=1}^{r} a_i \cdot (x \cdot f(a^i)) + [x, a_i] \cdot f(a^i)$$

Replace $x \cdot f(a^i) = f([x, a^i]) + a^i_{\mathfrak{k}} \cdot f(x)$ by 1-cocycle eq^n,

we get: $$\lambda (x \cdot m) = \underbrace{\sum_{i=1}^{r} a_i\, a^i \cdot f(x)}_{i=1} + \sum_{i=1}^{r} a_i \cdot f([x, a^i]) + [x, a_i] \cdot f(a^i)$$

1^{st} term $= C_M \cdot f(x) = \lambda f(x)$ since $C_M = \lambda \cdot Id_M$.

2^{nd} term is clearly 0 for $x \in \mathfrak{or}$, since $a_i, a^i \in \mathfrak{h}$ and $\mathfrak{g} \simeq \mathfrak{or} \oplus \mathfrak{h}$ as Lie algebras. For $x \in \mathfrak{h}$, we can proceed as in the proof of Prop. §2 of Lecture 14, or see that this term is obtained from applying the map

$\mathfrak{g} \otimes \mathfrak{g} \xrightarrow[\mathrm{id}\otimes f]{} \mathfrak{g} \otimes M \xrightarrow[\text{action map}]{} M$, to

$[x \otimes 1 + 1 \otimes x, \underbrace{\sum_{i=1}^{r} a_i \otimes a^i}_{\text{Casimir tensor } \Omega_M}]$ which is zero by Prop. 2 of Lecture 14. □

4. Corollary.- (a) If $\mathfrak{g}$ is finite-dim'l. semisimple over k (alg. closed of char. 0), then $\mathrm{Rep}_{f.d.}(\mathfrak{g})$ is semisimple.

(b) If $\mathfrak{g}$ is a finite-dim'l reductive Lie algebra, then

$$\mathfrak{g} \simeq \mathfrak{g}_{s.s.} \oplus Z(\mathfrak{g}) \text{ as Lie algebras.}$$

Proof.- (b): Consider $\mathfrak{g} \circlearrowright \mathfrak{g}$ via adjoint. As $Z(\mathfrak{g}) = \mathrm{Ker}(\mathrm{ad})$, we get an action of $\mathfrak{g}/Z(\mathfrak{g}) =: \mathfrak{g}_{s.s.}$ on $\mathfrak{g}$. By complete reducibility theorem $Z(\mathfrak{g}) \subset \mathfrak{g}$ sub-$\mathfrak{g}_{s.s.}$-representation admits a complementary subrepn.:

$$\mathfrak{g} \simeq Z(\mathfrak{g}) \oplus I \quad \text{as } \mathfrak{g}_{s.s.}\text{-representations.}$$

Moreover $I \simeq \mathfrak{g}/Z(\mathfrak{g}) = \mathfrak{g}_{s.s.}$ and the claim is proved. □

5. Theorem.- [$\mathfrak{g}$: f.d. s-s. and $k = \bar{k}$ of char 0].

Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite-dim'l representation. Let $x \in \mathfrak{g}$ and $x_S, x_N \in \mathfrak{g}$ be its semisimple / nilpotent components (Lecture 13, §4). Then $\rho(x) = \rho(x_S) + \rho(x_N)$ is the Jordan-Chevalley decomposition of $\rho(x) \in \mathfrak{gl}(V)$.

Proof.- We begin by assuming that V is irreducible (by complete reducibility theorem, this is sufficient), and moreover that ρ is faithful. The case of general ρ will be reduced to faithful representations as we did in the proof of Theorem 3 above.

Thus $\rho : \mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$, infact, since $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, $Im(\rho) \subset \mathfrak{sl}(V)$ where $\mathfrak{sl}(V) \subset \mathfrak{gl}(V)$ consists of operators of trace 0.

Let $\mathcal{N} := \{X \in \mathfrak{sl}(V) : [X, \rho(x)] \in Im(\rho),\ \forall x \in \mathfrak{g}\} \subset \mathfrak{sl}(V)$.
(normalizer of $\rho(\mathfrak{g}) \subset \mathfrak{sl}(V)$)

Then $\rho(\mathfrak{g}) \subset \mathcal{N}$ is a semisimple ideal, and hence (Lecture 13, §2)

$\mathcal{N} \simeq \rho(\mathfrak{g}) \oplus I$ as Lie algebras.

Claim: $I = \{0\}$.

[Pf.- For $a \in I \subset \mathfrak{sl}(V)$, $[a, \rho(x)] = 0 \quad \forall x \in \mathfrak{g}$ - i.e., $a : V \to V$ is a $\mathfrak{g}$-intertwiner. As V is irreducible, by Schur's Lemma $a = \mu \cdot Id_V$. But $Tr(a) = 0$, hence $\mu = 0$ i.e., $a = 0$. □]

Now let $x \in \mathfrak{g}$ and $\rho(x) = S + N$ be its J-C decomposition (as operators on V). Recall (Lecture 11, §4) that

$$ad(\rho(x)) = ad(S) + ad(N) : \mathfrak{sl}(V) \to \mathfrak{sl}(V)$$

is the J-C decomposition - and $ad(S)$, $ad(N)$ are polynomials in $ad(\rho(x))$. This implies that $[S, \rho(y)]$, $[N, \rho(y)] \in Im(\rho)$ and hence $S, N \in Im(\rho)$. Since ρ is injective

$\exists!$ $a_S, a_N \in \mathfrak{g}$ so that $\rho(a_S) = S$ and $\rho(a_N) = N$.

So, we have $[a_S, a_N] = 0$, $x = a_S + a_N$ and $\rho(x) = \rho(a_S) + \rho(a_N)$ the J-C decomposition of $\rho(x)$.

Since ρ is a representation, $\mathrm{ad}_{\mathfrak{sl}(V)} \circ \rho = \rho \circ \mathrm{ad}_{\mathfrak{g}}$ and injectivity of ρ implies $\mathrm{ad}(x) = \mathrm{ad}(a_S) + \mathrm{ad}(a_N)$ is the J.C.- decomposition in $\mathfrak{gl}(\mathfrak{g})$. By the uniqueness part of Lecture 13, §4, $a_S = x_S$ and $a_N = x_N$.

General case.- If $\rho : \mathfrak{g} \to \mathfrak{sl}(V)$ is not faithful, then

$$\mathfrak{g} \simeq \mathrm{Ker}(\rho) \oplus \mathfrak{h},$$ both semisimple and

$\rho|_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{sl}(V)$ is faithful. The theorem is thus true for $x \in \mathfrak{h}$, and trivially true for $x \in \mathrm{Ker}(\rho)$. Finally, if $[x, y] = 0$ in $\mathfrak{g}$, then $x + y = (x_S + y_S) + (x_N + y_N)$ is the J-C decomposition.

□