Lecture 17.

Representation theory of $sl_2(\mathbb{C})$

1. Recall. $sl_2(\mathbb{C})$ is 3-dimensional Lie algebra (over $\mathbb{C}$) on a basis $\{h, e, f\}$, with the Lie bracket given as follows:

$$[h,e] = 2e, \quad [h,f] = -2f, \text{ and } [e,f] = h.$$

Standard 2-dimensional representation $\mathbb{C}^2$:

$$h \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \text{ and } f \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

In Lecture 9, §3, we showed that $sl_2(\mathbb{C})$ is a simple Lie algebra.

Notice that in the ordered basis $\{e, h, f\}$, the matrices of $ad(h)$, $ad(e)$ and $ad(f)$ are:

$$ad(h) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad ad(e) = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad ad(f) = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

diagonal — nilpotent

Corollary to Theorem 3 and Theorem 5 of Lecture 16

(1) $\mathrm{Rep}_{f.d.}(sl_2(\mathbb{C}))$, the category of finite-dim'l representations of $sl_2(\mathbb{C})$, is semisimple.

(2) If $\rho : sl_2(\mathbb{C}) \to gl(V)$ is a finite-dimensional representation then $\rho(h)$ is semisimple, and $\rho(e)$, $\rho(f)$ are nilpotent

2. Classification of finite-dimensional, irreducible representations of $sl_2(\mathbb{C})$.-

For each $n \in \mathbb{Z}_{\geq 0}$, let L_n denote $(n+1)$-dimensional $\mathbb{C}$-vector space, with sl_2-action given in a fixed basis $\{\Omega_0, \ldots, \Omega_n\}$ as:

$$h \cdot \Omega_k = (n-2k)\Omega_k$$

$$e \cdot \Omega_k = (n-k+1)\Omega_{k-1}, \quad f \cdot \Omega_k = (k+1)\Omega_{k+1}$$

$$[\; e \cdot \Omega_0 = 0 \quad \text{and} \quad f \cdot \Omega_n = 0 \;].$$

Theorem.- (a) For each $n \in \mathbb{Z}_{\geq 0}$, L_n is an irreducible $(n+1)$-dim'l representation of $sl_2(\mathbb{C})$.

(b) If $V \in \mathrm{Rep}_{f.d.}(sl_2(\mathbb{C}))$ is irreducible, then

$$V \simeq L_n, \quad n = \dim(V) - 1.$$

Proof.- (a) We begin by verifying the defining relations of $sl_2(\mathbb{C})$ for the operations on L_n defined above:

- $he = e(h+2)$: evaluated on Ω_k $(0 \leq k \leq n)$ gives

$$\text{L.H.S.} = (n-k+1)\, h \cdot \Omega_{k-1} = (n-k+1)(n-2k+2)\Omega_{k-1}$$

$$\text{R.H.S.} = e\,(n-2k+2)\Omega_k = (n-k+1)(n-2k+2)\Omega_{k-1} \;\checkmark$$

[similar calculation for $hf = f(h-2)$.]

- $ef - fe = h$: $(ef - fe)\Omega_k = (k+1)\, e \cdot \Omega_{k+1} - (n-k+1) f \Omega_{k-1}$

$$= \big((k+1)(n-k) - (n-k+1)\cdot k\big)\Omega_k$$

$$= (n-2k)\Omega_k = h \cdot \Omega_k \quad \checkmark$$

Now we show that L_n is irreducible. So, if $\{0\} \neq W \subset L_n$ is a non-zero subrepresentation, we have to show that $W = L_n$. Let $w = \sum_{k=0}^{n} c_k \Omega_k \neq 0$ be in W. Then

$$h^r \cdot w = \sum_{k=0}^{n} c_k (n-2k)^r \Omega_k \in W \quad \text{for all } r \geq 0.$$

By van der ~~Warden~~ Monde trick, we conclude that $c_k \neq 0 \Rightarrow \Omega_k \in W$.

So, there is some $0 \leq k \leq n$, such that $\Omega_k \in W$. But then

$$e^k \cdot \Omega_k = (n-k+1)(n-k+2) \cdots n \cdot \Omega_0 \in \Omega \text{ and hence } \Omega_j = \frac{f^j}{j!} \Omega_0 \in W$$

for every $0 \leq j \leq n$. ~~an~~ Thus, $W = L_n$ as claimed.

(b) Now let $\rho : sl_2(\mathbb{C}) \to gl(V)$ be a finite-dim'l, irred. representation. Let $0 \neq w \in V$ be an eigenvector for $\rho(h)$ - i.e, $h \cdot w = \mu w$ $(\mu \in \mathbb{C})$.

Now $h \cdot e^r = e^r (h+2r)$ implies $e^r \cdot w$ is an eigenvector for $\rho(h)$ with eigenvalue $\mu + 2r$. By finite-dimensionality of V, there is some $r \geq 1$ s.t. $e^{r-1} \cdot w \neq 0$ and $e^r \cdot w = 0$.

Set $v_0 = e^{r-1} \cdot w$ so that $e \cdot v_0 = 0$

$\lambda = \mu + 2(r-1)$ $\quad h \cdot v_0 = \lambda \cdot v_0$.

$(\lambda \in \mathbb{C})$

Define $v_j = \frac{f^j}{j!} v_0$. Then $f \cdot v_j = (j+1) v_{j+1}$

$h \cdot v_j = (\lambda - 2j) v_j$

By dimension reasons, $\exists\, N \geq 0$ s.t. $v_N \neq 0$ and $v_{N+1} = 0$.

Claim.- $e \cdot f^{(n)} = f^{(n)} e + f^{(n-1)} (h-n+1)$ - (*)

Pf.- $[e, f^n] = \sum_{i=1}^{n} f^{n-i} [e,f] f^{i-1} = \sum_{i=1}^{n} f^{n-i} h f^{i-1}$

$= \sum_{i=1}^{n} f^{n-1} (h - 2(i-1))$ using $hf = f(h-2)$ repeatedly

$= f^{n-1} \cdot \left(nh - 2\frac{n(n-1)}{2} \right)$

$\Rightarrow e \cdot f^{(n)} = f^{(n)} \cdot e + f^{(n-1)} \cdot (h-n+1)$; where the divided power notation $x^{(n)} = \frac{x^n}{n!}$ is used. □

Continuing with the proof of the theorem. the claim implies, $\forall\, l \geq 0$,

$e \cdot v_l = e \cdot f^{(l)} \cdot v_0 = \left(f^{(l)} e + f^{(l-1)} (h - l + 1) \right) \cdot v_0$

$= (\lambda - l + 1)\, v_{l-1}$.

Taking $l = N+1$ in this equation gives ($v_{N+1} = 0$)

$0 = e \cdot v_{N+1} = (\lambda - N)\, v_N \quad \Rightarrow \quad \lambda = N \in \mathbb{Z}_{\geq 0}$

and $\text{Span}\{v_0, \ldots, v_N\} \subset V$ is a non-zero subrepresentation, hence $= V$ by irreducibility. Thus $N = \dim(V) - 1 = n$ and

$v_j \mapsto \Omega_j$ is an isomorphism of $\mathfrak{sl}_2$-repns. $V \xrightarrow{\sim} L_n$. □