Lecture 18

1. Recall - we are studying finite-dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$.

We have already proved the following theorem.-

- Every $V \in \mathrm{Rep}_{f.d.}(\mathfrak{sl}_2(\mathbb{C}))$ is semisimple.
- $\mathrm{Irred}_{f.d.}(\mathfrak{sl}_2(\mathbb{C}))$ = set of isomorphism classes of irreducible, finite-dim'l $\mathfrak{sl}_2$-representations

$$\longleftrightarrow \quad \mathbb{Z}_{\geq 0}$$

$$L_n \longleftrightarrow n \quad , \text{ where } L_n \text{ is } (n+1)\text{-dim'l}$$

(irred.) $\mathfrak{sl}_2$-representation, given in a basis $\{\Omega_0, \dots, \Omega_n\}$:

$$h \cdot \Omega_k = (n-2k)\Omega_k \quad , \quad e \cdot \Omega_k = (n-k+1)\Omega_{k-1}$$

$$f \cdot \Omega_k = (k+1)\Omega_{k+1} \qquad (\Omega_{-1} = 0 = \Omega_{n+1})$$

More geometrically, $\mathfrak{sl}_2$ acts on $\mathbb{C}[x,y]$ = homogeneous coordinate ring of $\mathbb{P}^1(\mathbb{C})$

via: $h = x\partial_x - y\partial_y$ (= deg in x − deg. in y)

$e = x\partial_y$ (decreases deg in y by 1, increases deg in x by 1)

$f = y\partial_x$ (other way around)

[Check: $ef - fe = x\partial_y\, y\partial_x - y\partial_x\, x\partial_y$

$= x(y\partial_y + 1)\partial_x - y(x\partial_x + 1)\partial_y \qquad (\partial_x x = x\partial_x + 1)$

$= x\partial_x - y\partial_y = h$ ✓]

Then $L_n \simeq \mathbb{C}[x,y]_{\deg = n}$

$$\Omega_0 \longmapsto \frac{x^n}{n!}$$
$$\vdots$$
$$\Omega_k \longmapsto \frac{x^{n-k}}{(n-k)!}\,\frac{y^k}{k!}$$

[Ex: check that this map is an sl_2-intertwiner.]

2. Some consequences of this classification & complete reducibility thm.-

Let $V \in \mathrm{Rep}_{f.d.}(sl_2(\mathbb{C}))$. As h-action on V is diagonalizable with integer eigenvalues, we can write

$$V = \bigoplus_{\mu \in \mathbb{Z}} V[\mu], \quad \text{where } V[\mu] = \{v \in V : h \cdot v = \mu v\}.$$

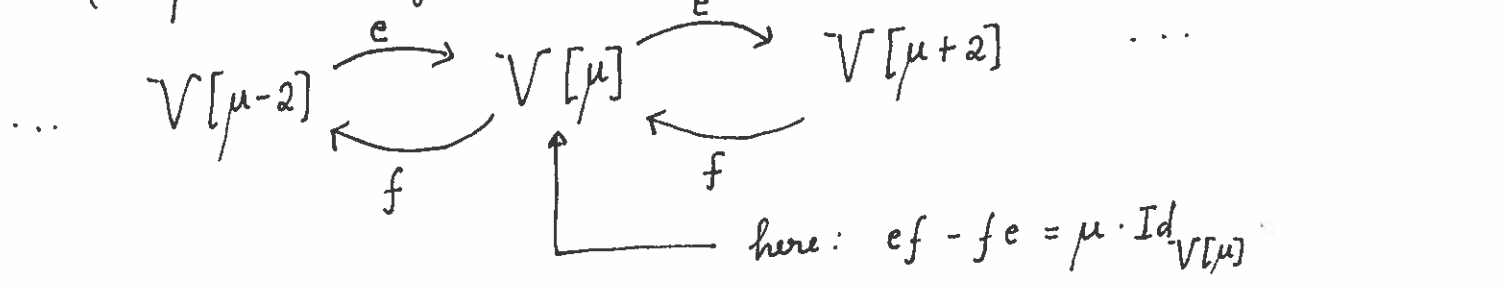
$V[\mu]$ is often called μ-weight space, $\mu \in \mathbb{Z}$ s.t. $V[\mu] \neq \{0\}$ is called a weight of V

$$P(V) := \{\mu : V[\mu] \neq \{0\}\} \subset \mathbb{Z}$$

the set of weights of V.

e.g. $P(L_n) = \{-n, -n+2, \dots, n-2, n\}$

The action of e on V (resp. f) has the effect of increasing (resp. decreasing) weights by 2. That is,



For this reason, e is often called the raising operator.
(f) (lowering)

(a) $\mathrm{Ker}(e) = \{v \in V : e \cdot v = 0\} \subset V$ is called the space of singular vectors. If $V \simeq \bigoplus_{n=0}^{\infty} L_n^{\oplus d_n}$ ($d_0, d_1, \ldots \in \mathbb{Z}_{\geq 0}$ only finitely many non-zero.)

then

$$\sum_{n=0}^{\infty} d_n = \text{number of irreducible summands in } V$$
$$= \dim\left(\mathrm{Ker}(e)\right).$$

(b) h preserves the space of singular vectors - since

$$e \cdot v = 0 \quad \Rightarrow \quad e \cdot (h(v)) = h \cdot (e \cdot v) - 2\, e \cdot v$$
$$= 0.$$

Thus if $\mathrm{Ker}(e) = \bigoplus_{\mu \in \mathbb{Z}} \mathrm{Ker}(e)[\mu]$, then:

$$\boxed{\mathrm{Ker}(e)[\mu] \neq 0 \Rightarrow \mu \in \mathbb{Z}_{\geq 0}}$$ "dominant weight"

$\forall\ 0 \neq v \in \mathrm{Ker}(e)[\mu]$, $\exists$ $\mathfrak{sl}_2$-intertwiner

$$\varphi_v : L_\mu \longrightarrow V$$
$$\Omega_0^{(\mu)} \longmapsto v$$

This, more concretely realizes the decomposition $V \simeq \bigoplus_{n=0}^{\infty} L_n^{\oplus d_n}$

as: $d_n = \dim \mathrm{Hom}_{\mathfrak{sl}_2}(L_n, V)$
$= \dim\left(\mathrm{Ker}(e)[n]\right)$, and

$$\boxed{\bigoplus_{n=0}^{\infty} L_n \otimes \operatorname{Hom}_{sl_2}(L_n, V) \xrightarrow{\sim} V}$$ as sl_2-representations, where the action of sl_2 on $L_n \otimes \operatorname{Hom}_{sl_2}(L_n, V)$ is only on the first tensor factor. The space $\operatorname{Hom}_{sl_2}(L_n, V)$ is called the multiplicity space.

(c) Weight symmetry - The set of weights $P(V) \subset \mathbb{Z}$ and their multiplicity $m_\mu := \dim V[\mu]$, $\mu \in P(V)$ ($m_\mu \in \mathbb{Z}_{\geq 1}$)

obey certain inequalities - which follow from

$e : V[\mu] \longrightarrow V[\mu+2]$ is injective for $\mu \leq -1$.
surjective for $\mu \geq 0$.

$f : V[\mu] \longrightarrow V[\mu-2]$ is injective for $\mu \geq 1$.
surjective for $\mu \leq 0$.

Moreover, $\boxed{\dim V[\mu] = \dim V[-\mu]}$ Weyl symmetry.

Thus, if $P(V) \subset \{-N, -N+1, \ldots, N-1, N\}$, then

mult $m_{-N}, m_{-N+1}, \ldots, m_{N-1}, m_N \in \mathbb{Z}_{\geq 0}$

then
$$\begin{array}{ccccccc} m_N & \leq & m_{N-2} & \leq & \cdots & \leq & m_{0 \text{ or } 1} \equiv N \pmod 2 \\ \| & & \| & & \cdots & & \| \\ m_{-N} & \leq & m_{-N+2} & \leq & \cdots & \leq & m_{0 \text{ or } -1} \equiv N \pmod 2 \end{array}$$

3. The following facts are worth recording for future use:

Prop - Let V be a finite-dim'l sl_2-representation.

(a) If $\exists\ 0 \neq v \in V$ s.t. $e \cdot v = 0$ then $\mu \in \mathbb{Z}_{\geq 0}$
$h \cdot v = \mu v$

In this case $f^{\mu+1} \cdot v = 0$.

(b) If $P(V) \subset 2 \cdot \mathbb{Z}$ and $\dim V[0] = 1$
(resp. $2 \cdot \mathbb{Z} + 1$) (resp. $\dim V[1] = \dim V[-1] = 1$)

then V is irreducible.

(Proof follows from (b) and (c) of the previous section.).

4. Weyl symmetry. The numerical identity $\dim V[\mu] = \dim V[-\mu]$ can be obtained from a (vector space) automorphism

$S_V \in \underset{\text{v.s.}}{\text{Aut}}(V)$, which reverses the signs of the weights "simple reflection"

$\mu \mapsto -\mu$.

Definition - Let $\rho : sl_2(\mathbb{C}) \to gl(V)$ be a finite-dimensional representation.

$$S_V := \exp(\rho(e)) \exp(-\rho(f)) \exp(\rho(e)) \quad - \quad (*)$$

[triple product formula - due to Jacques Tits.]

here, for a nilpotent operator $X : V \to V$, $\exp(X) := Id + X + \frac{X^2}{2!} + \cdots$

is a well-defined automorphism of V.

Properties of exponential

(i) If $ab = ba$ then $\exp(a)\cdot\exp(b) = \exp(a+b)$ in $\mathbb{Q}[\![a,b]\!]$

(formal power series)

(ii) Let B be an arbitrary algebra over $\mathbb{C}$.

(not necessarily unital, commutative or associative)

$D : B \to B$ be a <u>nilpotent derivation</u>.

Then, $\exp(D) : B \longrightarrow B$ is an algebra isomorphism.

(iii) Let $\rho : \mathfrak{a} \to \mathfrak{gl}(V)$ be a finite-dim'l representation of a Lie algebra $\mathfrak{a}$ (over $\mathbb{Q}$ at least).

Let $a \in \mathfrak{a}$ be such that $ad(a) : \mathfrak{a} \to \mathfrak{a}$ and $\rho(a) : V \to V$ are nilpotent.

Then, for every $b \in \mathfrak{a}$

$$\rho\left(\exp(ad(a))\cdot b\right) = \exp(\rho(a))\cdot\rho(b)\cdot\exp(-\rho(a))$$

[Proof is left as an easy exercise.]

<u>Cor.</u> $\forall\ V \in \mathrm{Rep}_{f.d.}(\mathfrak{sl}_2(\mathbb{C}))$, we have a linear automorphism

$\mathcal{S}_V \in \mathrm{Aut}_{v.s.}(V)$ defined in (*) on the previous page.

s.t. (i) $\forall\ f : V \to W$ $\mathfrak{g}$-intertwiner

$$f \circ \mathcal{S}_V = \mathcal{S}_W \circ f$$

(ii) For $V = \mathfrak{sl}_2(\mathbb{C}) = \mathfrak{g}$, adjoint representation

$$\delta_{\mathfrak{g}} \in \operatorname{Aut}_{\text{Lie Alg}}(\mathfrak{sl}_2(\mathbb{C})).$$

(iii) If $\rho : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V)$ is a finite-dim'l repn,

then $\delta_V(\rho(x) \cdot v) = \rho(\delta_{\mathfrak{g}}(x)) \cdot \delta_V(x)$, $\forall$ $x \in \mathfrak{sl}_2(\mathbb{C})$, $v \in V$.