# Lecture 19

**1. Recall** - $\mathrm{Rep}_{f.d.}(\mathfrak{sl}_2(\mathbb{C}))$ = category of finite-dim'l representations of $\mathfrak{sl}_2(\mathbb{C})$ is semisimple, with irreducible representations being labelled by $\mathbb{Z}_{\geq 0}$. For each $n \in \mathbb{Z}_{\geq 0}$, $L_n$ is $(n+1)$-dim'l $\mathbb{C}$-vector space on a basis $\{m_0, \dots, m_n\}$ in which $\mathfrak{sl}_2$-action is given as:

$$h \cdot m_k = (n-2k) m_k \; ; \quad e \cdot m_k = (n-k+1) m_{k-1} \; , \; f \cdot m_k = (k+1) m_{k+1}$$

$$[\, m_{-1} = 0 = m_{n+1} \,].$$

**Some terms and notations** $V \in \mathrm{Rep}_{f.d.}(\mathfrak{sl}_2(\mathbb{C}))$,

Weight space decomposition

$$V \overset{\downarrow}{=} \bigoplus_{\mu \in \mathbb{Z}} V[\mu]$$

(vector space direct sum)

$$V[\mu] := \{ v \in V : h \cdot v = \mu v \}$$

$\mu$-weight space

$P(V) =$ set of weights of $V = \{ \mu : V[\mu] \neq \{0\} \} \overset{\text{finite}}{\subset} \mathbb{Z}$

**Weyl symmetry** : $\forall \mu \in P(V)$, $-\mu \in P(V)$ and $\dim V[\mu] = \dim V[-\mu]$.

**2.** In the last lecture we were giving an a-priori explanation of Weyl symmetry. We defined

$$\rho : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V) \quad \rightsquigarrow \quad s_V = \exp(\rho(e)) \exp(-\rho(f)) \exp(\rho(e))$$

f.d. repn.

$$\in \mathrm{Aut}_{\text{v.s.}}(V)$$

$$\left[ \exp(A) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n \text{ , for a nilpotent } A \in \mathrm{End}_{\text{v.s.}}(V) \right]$$

Using elementary properties of exponential series, we showed:

**Proposition** - (i) When $V = L_2 = sl_2(\mathbb{C})$ adjoint representation

$$\sigma := \mathcal{S}_{sl_2\text{-adj.}} \in \underset{\text{LieAlg}}{\text{Aut}}(sl_2(\mathbb{C}))$$

(ii) For every $sl_2$- intertwiner $f : V \to W$, $f \circ \mathcal{S}_V = \mathcal{S}_W \circ f$.

(iii) For $V \in \text{Rep}_{\text{f.d.}}(sl_2(\mathbb{C}))$, $\rho : sl_2(\mathbb{C}) \to \mathfrak{gl}(V)$, $x \in sl_2(\mathbb{C})$ and $v \in V$, we have

$$\mathcal{S}_V(\rho(x)(v)) = \rho(\sigma(x)) \cdot (\mathcal{S}_V(v)).$$

## 3. Two more properties of $\mathcal{S}_V$ operations -

(a) $$\mathcal{S}_{V_1 \oplus V_2} = \mathcal{S}_{V_1} \oplus \mathcal{S}_{V_2} = \left[\begin{array}{c|c} \mathcal{S}_{V_1} & 0 \\ \hline 0 & \mathcal{S}_{V_2} \end{array}\right]$$

(this is clear from definitions)

(b) $$\mathcal{S}_{V \otimes W} = \mathcal{S}_V \otimes \mathcal{S}_W$$ - i.e. $\mathcal{S}$ is "group-like".

(Proof - check that if $a \in H$ (bialgebra) is primitive i.e. $\Delta(a) = a \otimes 1 + 1 \otimes a$, then

$$\Delta(\exp(a)) = \exp(a) \otimes \exp(a).$$ )

(c) Let $V = L_1 = \mathbb{C}^2$ the standard vector representation.

Then $$\mathcal{S}_{L_1} = \exp\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \cdot \exp\left(\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}\right) \cdot \exp\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{"reflection"}$$

i.e. in the basis $\{m_0, m_1\}$ : $\mathcal{S}_{L_1}$ : $m_0 \mapsto -m_1$, $m_1 \mapsto m_0$

4. Theorem.- (a) For any $V \in \mathrm{Rep}_{f.d.}(\mathfrak{sl}_2(\mathbb{C}))$ and $\mu \in \mathbb{Z}$;

$$\mathcal{S}_V : V[\mu] \xrightarrow{\sim} V[-\mu]$$

(b) For each $n \in \mathbb{Z}_{\geq 0}$, $\mathcal{S}_{L_n} : m_k \mapsto (-1)^{n-k} m_{n-k}$.

Proof.- We begin by computing $\sigma = \mathcal{S}_{\mathfrak{sl}_2\text{-adj. repn}} \in \mathrm{Aut}_{\text{Lie}}(\mathfrak{sl}_2(\mathbb{C}))$.

This (also the computation of (b)) can be done in two ways:

I. Compute explicitly 3×3 matrices $\exp(\mathrm{ad}(e))$ and $\exp(\mathrm{ad}(f))$:

$$\mathrm{ad}(e) = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathrm{ad}(e)^2 = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{higher} = 0$$

$$\mathrm{ad}(f) = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \qquad \mathrm{ad}(f)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}$$

$$\exp(\mathrm{ad}(e)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$= \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \exp(-\mathrm{ad}(f)) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix}$$

$$\Rightarrow \sigma = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

i.e. $\sigma$ : $e \longmapsto -f$

$h \longmapsto -h$

$f \longmapsto -e$

II Realize $L_2 \hookrightarrow L_1 \otimes L_1$

$$m_0^{(2)} \longmapsto m_0^{(1)} \otimes m_0^{(1)}$$

$$m_1^{(2)} = f \cdot m_0^{(2)} \longmapsto m_0^{(1)} \otimes m_1^{(1)} + m_1^{(1)} \otimes m_0^{(1)}$$

$$m_2^{(2)} = \frac{f^2}{2!} \cdot m_0^{(2)} \longmapsto m_1^{(1)} \otimes m_1^{(1)}$$

we know $\rho_{L_1}$ : $m_0 \longmapsto -m_1$ from §3 (c) above and

$m_1 \longmapsto m_0$

$\rho_{L_1 \otimes L_1} = \rho_{L_1} \otimes \rho_{L_1}$ gives $\rho_{L_2}$ :

$$m_0^{(2)} \longmapsto m_2^{(2)}$$

$$m_1^{(2)} \longmapsto -m_1^{(2)}$$

$$m_2^{(2)} \longmapsto m_2^{(2)}$$

Now $L_2 \simeq \mathfrak{sl}_2(\mathbb{C})$

$m_0 \longleftrightarrow e$

$m_1 = f \cdot m_0 \longleftrightarrow -h$ (ad(f))

$m_2 = \frac{f^2}{2!} \cdot m_0 \longleftrightarrow -f$ ($\frac{1}{2}$ ad(f))

Hence, $\sigma$: $e \mapsto -f$

$h \mapsto -h$

$f \mapsto -e$

♦

We can now prove (a) : $\mathcal{S}_V : V[\mu] \longrightarrow V[-\mu]$. Let $v \in V[\mu]$.

Then, $\sigma(h)\cdot \mathcal{S}_V(v) = \mathcal{S}_V(h\cdot v) = \mathcal{S}_V(\mu v) = \mu\, \mathcal{S}_V(v)$

↑ by (3) of Prop. §2 page 2 above

Since $\sigma(h) = -h$, we get $h\cdot \mathcal{S}_V(v) = -\mu\, \mathcal{S}_V(v)$, i.e., $\mathcal{S}_V(v) \in V[-\mu]$ as claimed.

The computation for (b) can be carried out in two ways - as before :

I. Verify directly that $\mathcal{S}_V(m_0) = (-1)^n m_n \qquad (V = L_n)$

(recall $\{m_0, \dots, m_n\}$ is a basis of $L_n$.)

$$\mathcal{S}_V(m_0) = \exp(e)\exp(-f)\exp(e)\cdot m_0 \qquad (e\cdot m_0 = 0)$$
$$= \exp(e)\exp(-f)\cdot m_0$$
$$= \left(1 + e + \tfrac{1}{2}e^2 + \cdots\right)\left(m_0 - m_1 + m_2 - \dots + (-1)^n m_n\right)$$

We know $\mathcal{S}_V(m_0) \in V[-n] = \mathbb{C}\cdot m_n$ ; and $m_n \notin \text{Image}(e)$

So, $\mathcal{S}_V(m_0) = (-1)^n m_n$.

II. Use $L_n \hookrightarrow \underbrace{L_1 \otimes \cdots \otimes L_1}_{n\text{-fold}}$ $\Rightarrow \mathcal{S}_V(m_0) = (-1)^n m_n$

$$m_0^{(n)} \longmapsto m_0^{(1)} \otimes \cdots \otimes m_0^{(1)}$$
$$(-1)^n\, m_n^{(1)} \otimes \cdots \otimes m_1^{(1)}$$

$\mathcal{S}_{L_1^{\otimes n}} = \underbrace{\mathcal{S}_{L_1} \otimes \cdots \otimes \mathcal{S}_{L_1}}_{n\text{-fold}}$

$$m_n^{(n)} = \frac{f^n}{n!}\cdot m_0^{(n)} \longmapsto m_1^{(1)} \otimes \cdots \otimes m_1^{(1)} \quad [\text{Check this.}]$$

Finally $\delta_V(m_\ell) = \delta_V\left(\frac{f^\ell}{\ell!}\cdot m_0\right) = \sigma\left(\frac{f^\ell}{\ell!}\right)\cdot(-1)^n m_n$

$$= (-1)^n (-1)^\ell \frac{e^\ell}{\ell!}\cdot m_n = (-1)^{n-\ell} m_{n-\ell}\,. \qquad \square$$

5. Remark.- The result stated in Prop. §2, page 2 above can be rephrased as follows. Let $F : \mathrm{Rep}_{f.d.}(sl_2(\mathbb{C})) \longrightarrow \mathrm{Vect}_{f.d.}$ be the forgetful functor. Let $\mathrm{End}(F) =: \hat{\mathcal{U}}$ be the set of all natural transformations $\eta : F \Rightarrow F$. Recall - this is the data of $\left\{ \eta_V : V \to V \text{ linear map} \right\}_{V \in \mathrm{Rep}_{f.d.}(sl_2)}$ s.t. $\forall\ f : V \to W$ in $\mathrm{Rep}_{f.d.}(sl_2)$ we have $f \circ \eta_V = \eta_W \circ f$.

$\hat{\mathcal{U}}$ has the structure of $\mathbb{C}$-vector space:

$$(\lambda_1 \eta_1 + \lambda_2 \eta_2)_V = \lambda_1 \eta_{1,V} + \lambda_2 \eta_{2,V}$$

$$\forall\ \lambda_1, \lambda_2 \in \mathbb{C}\ ;\ \eta_1, \eta_2 : F \Rightarrow F\ ;\ V \in \mathrm{Rep}_{f.d.}(sl_2)$$

and an associative, unital algebra (unit $\mathbb{1}_V = Id_V$) with multiplication = composition.

$\hat{\mathcal{U}}$ is called the completion of $\mathcal{U}(sl_2)$ with respect to the category of finite-dim'l representations.

Note: there is a natural algebra hom.

$$\mathcal{U}(sl_2) \longrightarrow \hat{\mathcal{U}} \quad \text{given as follows}$$

$x \in U(sl_2) \rightsquigarrow \eta(x) : F \Rightarrow F$, for $\rho_V : sl_2 \to gl(V)$ in $\mathrm{Rep}_{f.d.}(sl_2(\mathbb{C}))$

denote by the same letter $\rho_V : U(sl_2) \to \mathrm{End}(V)$.

Then, $\eta(x)_V = \rho_V(x) : V \to V$ linear map which commutes with all $g = sl_2$-intertwiners.

[Homework Problem] $U(sl_2) \longrightarrow \hat{U}$ is injective.

i.e. $\forall\ 0 \neq x \in U(sl_2)$, $\exists\ \rho : U(sl_2) \to \mathrm{End}(V)$ in $\mathrm{Rep}_{f.d.}(sl_2)$

s.t. $\rho(x) \neq 0$.

$S = \exp(e)\exp(-f)\exp(e)$ is a well-defined element of $\hat{U}$, though not in $U(sl_2)$.

Easy exercise - 1 is the only group-like element (i.e. $\Delta(g) = g \otimes g$) in $U(g)$.