# Lecture 20

## 1. Characters of finite-dimensional, $\mathfrak{sl}_2(\mathbb{C})$-representations -

For $V \in \mathrm{Rep}_{f.d.}(\mathfrak{sl}_2(\mathbb{C}))$, define $\chi_V(z) \in \mathbb{Z}[z^{\pm 1}]$ by:

$$\chi_V(z) = \sum_{\mu \in \mathbb{Z}} (\dim V[\mu]) \cdot z^{\mu}$$

"generating function for multiplicities of weight spaces of $V$".

e.g. $V = L_n$ irreducible $(n+1)$-dim'l representation

$$\chi_{L_n}(z) = z^n + z^{n-2} + \cdots + z^{-n+2} + z^{-n}$$

$$= \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}}$$

**Proposition** · (i) $\chi_{V_1 \oplus V_2} = \chi_{V_1} + \chi_{V_2}$

(ii) $\chi_V(z^{-1}) = \chi_V(z)$ (Weyl symmetry)

(iii) $\chi_{V_1 \otimes V_2}(z) = \chi_{V_1}(z) \cdot \chi_{V_2}(z)$

(Pf. of (iii) $(V_1 \otimes V_2)[\mu] = \bigoplus_{\mu_1 + \mu_2 = \mu} V_1[\mu_1] \otimes V_2[\mu_2]$ · )

## 2. Clebsch-Gordon rule -

$$L_m \otimes L_n \simeq L_{m+n} \oplus L_{m+n-2} \oplus \cdots \oplus L_{m+n-2(\mathrm{MIN}(m,n))}$$

Note: $\{\chi_{L_n}(z)\}_{n=0,1,\dots}$ is a basis of $\mathbb{C}[z,z^{-1}]^{\mathbb{Z}/2\mathbb{Z}}$

$\mathbb{Z}/2\mathbb{Z}$ acts by $\bar{1}: z \mapsto z^{-1}$.

and the problem of decomposing $V \in \mathrm{Rep}_{f.d.}(\mathfrak{sl}_2(\mathbb{C}))$ as a direct sum of irreducibles is equivalent to writing $\chi_V(z)$ in this basis:

$$V \simeq \bigoplus_{\mu \in \mathbb{Z}_{\geq 0}} L_\mu^{\oplus d_\mu} \iff \chi_V(z) = \sum_{\mu \in \mathbb{Z}_{\geq 0}} d_\mu \cdot \chi_{L_\mu}(z).$$

e.g. for $m \geq 1$, $L_m \otimes L_1 \simeq L_{m+1} \oplus L_{m-1}$, since (Pieri rule)

$$\begin{aligned}
&\left(z^m + z^{m-2} + \dots + z^{-m+2} + z^{-m}\right)\left(z + z^{-1}\right) \\
&= z^{m+1} + z^{m-1} + \dots + z^{-m+3} + z^{-m+1} \\
&\quad + z^{m-1} + z^{m-3} + \dots + z^{-m+1} + z^{-m-1} \\
&= \left(z^{m+1} + \dots + z^{-m-1}\right) + \left(z^{m-1} + \dots + z^{-m+1}\right).
\end{aligned}$$

Ex. Prove the Clebsch-Gordon rule using characters.

## 3. Verma modules and category $\mathcal{O}(\mathfrak{sl}_2(\mathbb{C}))$.

For any $\lambda \in \mathbb{C}$, the Verma module $M_\lambda$ is the representation of $\mathfrak{sl}_2(\mathbb{C})$ satisfying the following universal property:

(a) $\exists$ a non-zero vector $\Omega_0 \in M_\lambda$ s.t.

$$e \cdot \Omega_0 = 0 \quad \text{and} \quad h \cdot \Omega_0 = \lambda \Omega_0$$

(b) If $V$ is any $\mathfrak{sl}_2(\mathbb{C})$-representation, $v \in V$ s.t.

$$e \cdot v = 0 \quad \text{and} \quad h \cdot v = \lambda v,$$

then $\exists!$ $\mathfrak{sl}_2$-intertwiner $\varphi : M_\lambda \to V$

s.t. $\varphi(\Omega_0) = v$

Thus $\operatorname{Hom}_{\mathfrak{sl}_2}(M_\lambda, V) = \{ v \in V[\lambda] : e \cdot v = 0 \}$

"singular weight vectors of weight $\lambda$."

Construction - $M_\lambda := \mathcal{U}(\mathfrak{sl}_2(\mathbb{C})) \,/$ left ideal generated by $(e,\ h - \lambda \cdot 1)$

$\Omega_0 =$ equivalence class of $1 \in \mathcal{U}(\mathfrak{sl}_2(\mathbb{C}))$

Proof of the universal property: Let $V$ be an $\mathfrak{sl}_2$-representation and $v \in V$. Viewed as a $\mathcal{U}$-module ($\mathcal{U} = \mathcal{U}(\mathfrak{sl}_2(\mathbb{C}))$) we have $\operatorname{Hom}_{\mathcal{U}\text{-mod}}(\mathcal{U}, V) \simeq V$. So let

$$\begin{array}{ccc} \psi & & \psi \\ \varphi & \longmapsto & \varphi(1) \end{array}$$

$\varphi : \mathcal{U} \to V$ be the unique intertwiner s.t. $\varphi(1) = v$

Since $h \cdot v = \lambda \cdot v$, $e \cdot v = 0$, we get $\varphi(x) = 0$ $\forall\ x \in \mathcal{U} \cdot e + \mathcal{U} \cdot (h - \lambda)$

i.e. $\varphi : M_\lambda \to V$ is $\mathfrak{sl}_2$-intertwiner.

$\varphi(\Omega_0) = v$

□

More explicitly, by PBW theorem

$$U(\mathfrak{sl}_2) \simeq U^- \cdot U^0 \cdot U^+ \qquad U^\pm \text{ - poly's in } e/f$$
$$U^0 \text{ - " " } h$$

$U$ / left ideal generated by $e,\ h-\lambda\cdot 1$

gives a vector space iso. $M_\lambda \simeq U^-$. Thus, $M_\lambda$ has a basis $\left\{\Omega_\ell = \frac{f^\ell}{\ell!}\cdot\Omega_0\right\}_{\ell=0,1,2,\dots}$ in which $\mathfrak{sl}_2$-action is given by our usual formulae

$$h\cdot\Omega_\ell = (\lambda-2\ell)\,\Omega_\ell$$
$$f\cdot\Omega_\ell = (\ell+1)\,\Omega_{\ell+1}$$
$$e\cdot\Omega_\ell = (\lambda-\ell+1)\,\Omega_{\ell-1} \qquad (e\cdot\Omega_0 = 0).$$

Ex. (i) $M_\lambda$ is irreducible $\forall\ \lambda\in\mathbb{C}\setminus\mathbb{Z}_{\geq 0}$.

(ii) If $\lambda\in\mathbb{Z}_{\geq 0}$, we have a (non-split) short exact sequence

$$0 \to M_{-\lambda-2} \longrightarrow M_\lambda \longrightarrow L_\lambda \to 0$$

$L_\lambda$: $(\lambda+1)$-dim'l irreducible

$$\Omega_0^{(-\lambda-2)} \longmapsto \Omega_{\lambda+1}^{(\lambda)}$$

4. Category $\mathcal{O}(\mathfrak{sl}_2(\mathbb{C}))$ .- An $\mathfrak{sl}_2$-representation $V$ is in category $\mathcal{O}$ if (i) $h$-action on $V$ is semisimple, with finite-dim'l weight spaces:

$$V = \bigoplus_{\mu \in \mathbb{C}} V[\mu] \quad , \quad \dim V[\mu] < \infty \quad \forall \mu \in \mathbb{C}.$$

Let $P(V) \subset \mathbb{C}$ be the set of weights of $V$.

(ii) $\exists\ \lambda_1, \dots, \lambda_r \in \mathbb{C}$ s.t. $P(V) \subset \bigcup_{j=1}^{r} \lambda_j - 2\mathbb{Z}_{\geq 0}$.

A representation $V$ of $\mathfrak{sl}_2(\mathbb{C})$ is said to be a highest-weight representation of highest weight $\lambda \in \mathbb{C}$ if $\exists\ 0 \neq v \in V[\lambda]$ s.t. $e \cdot v = 0$ and $V = \mathcal{U}(\mathfrak{sl}_2) \cdot v$

$(h \cdot v = \lambda v)$

Remarks - (a) $M_\lambda \in \mathcal{O}(\mathfrak{sl}_2(\mathbb{C}))$ are "universal highest weight repns. of h.w. $\lambda$".

(b) If $V \in \mathcal{O}(\mathfrak{sl}_2(\mathbb{C}))$ is irreducible, then $V$ is necessarily highest weight. The converse is not true - e.g. if $\lambda \in \mathbb{Z}_{\geq 0}$, then $M_\lambda$ is a h.w. repn. but is not irreducible.