# Lecture 21

## Towards Cartan-Killing structure theorem

1. **Example of $\mathfrak{sl}_{n+1}(\mathbb{C})$.** - Let $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$, $n \in \mathbb{Z}_{\geq 1}$, the Lie algebra of traceless $(n+1)\times(n+1)$ matrices. [Notation: $E_{kl}$ for elementary matrices]

→ **Cartan subalgebra** $\mathfrak{h}$ = (abelian) Lie subalgebra of $\mathfrak{g}$, consisting of diagonal matrices.

Note: $\dim \mathfrak{h} = n$. $\{h_i = E_{ii} - E_{i+1,i+1}\}_{i=1,\dots,n}$ is a basis of $\mathfrak{h}$

Viewing $\mathfrak{g}$ as a representation of $\mathfrak{h}$ via adjoint, we have the notion of roots and root spaces:

$$\mathfrak{g}_\alpha := \{ x \in \mathfrak{g} : [h, x] = \alpha(h)\, x \ \forall\, h \in \mathfrak{h} \}$$

($\alpha \in \mathfrak{h}^*$). (**root space**)

$$R \subset \mathfrak{h}^* \setminus \{0\} \qquad R = \{ \alpha \in \mathfrak{h}^* \setminus \{0\} \text{ s.t. } \mathfrak{g}_\alpha \neq \{0\} \}$$

(**set of roots**)

Since $\operatorname{ad} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{n+1} \end{bmatrix} \cdot E_{kl} = (\lambda_k - \lambda_l) E_{kl}$

$(1 \leq k, l \leq n+1)$

$$R = \{ \alpha_{kl} : 1 \leq k, l \leq n+1,\ k \neq l \}, \text{ where } \alpha_{k,l} : \mathfrak{h} \longrightarrow \mathbb{C}$$

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{n+1} \end{bmatrix} \mapsto \lambda_k - \lambda_l$$

Then we obtain a finite set of vectors $R = \{\varepsilon_k - \varepsilon_\ell : 1 \le k, \ell \le n+1, k \ne \ell\}$ in a real Euclidean space $\mathbb{R}^{n+1}$ $(= (\mathbb{R}\text{-span of } \{h_1, \dots, h_n\})^* \subset \mathbb{R}^{n+1})$ called a root system in general.

Definition - Let $E$ be a real vector space, together with a positive-definite form $(\cdot,\cdot) : E \times E \to \mathbb{R}$.

For $0 \ne v \in E$, $s_v \in O(E) = \{g \in GL(E) : (gx, gy) = (x,y) \ \forall x, y \in E\}$ orthogonal group

is defined by:

$$\boxed{s_v(x) = x - \frac{2(x,v)}{(v,v)} \cdot v}$$

reflection in the hyperplane orthogonal to $v$.

A root system $R$ is a finite set of non-zero vectors in $E$ satisfying the following list of axioms:

(i) $R$ spans $E$. If $\alpha, \beta \in R$ are proportional, then $\beta = \pm\alpha$.

(ii) $\forall \alpha, \beta \in R$, $\dfrac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$

(iii) $\forall \alpha \in R$, $s_\alpha(R) = R$.

Ex. $R = \{\varepsilon_k - \varepsilon_\ell : 1 \le k, \ell \le n+1, k \ne \ell\} \subset \mathbb{R}^{n+1}$ w/ standard inner product

satisfies these axioms. The group generated by reflections $s_{\varepsilon_k - \varepsilon_\ell} =$ permutation $(k\ell) \in S_{n+1}$ is $S_{n+1} \circlearrowright E = \{\underline{x} \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i = 0\}$ called the Weyl group.

A base $\{\alpha_i\}_{i\in I}$ of a root system $R$ is a subset such that (i) $\{\alpha_i\}_{i\in I}$ is linearly independent.

(sign-coherence) (ii) $\forall\ \alpha \in R$, $\alpha = \sum_{i\in I} n_i \alpha_i$. Then, either $n_i \in \mathbb{Z}_{\geq 0}$ $\forall\ i\in I$ or $n_i \in \mathbb{Z}_{\leq 0}$ $\forall\ i\in I$.

Example.- $R = \{\varepsilon_k - \varepsilon_\ell : 1\leq k\neq \ell \leq n+1\}$ as before. Take $\Delta = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} : 1\leq i\leq n\}$. Then

if $k<\ell$, $\varepsilon_k - \varepsilon_\ell = \alpha_k + \cdots + \alpha_{\ell-1}$

if $k>\ell$, $\varepsilon_k - \varepsilon_\ell = -(\varepsilon_\ell - \varepsilon_k) = -(\alpha_\ell + \cdots + \alpha_{k-1})$

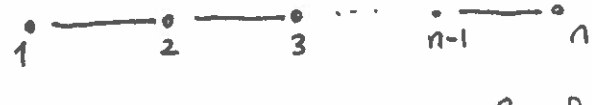
} all $\geq 0$ or all $\leq 0$

Cartan matrix of the pair $(R, \Delta = \{\alpha_i\}_{i\in I})$ is $I\times I$ integer matrix $A = (a_{ij})_{i,j\in I}$ where

$$a_{ij} = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \in \mathbb{Z}$$ by axiom (ii) of root systems

Example- in the choice of $\Delta$ of the example above

$$A = \begin{bmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{bmatrix}_{n\times n}$$




Dynkin diagram of $\mathfrak{sl}_{n+1}(\mathbb{C})$.

Cartan-Killing classification theorem gives a complete list of Cartan matrices / Dynkin diagrams associated to a simple Lie algebra.



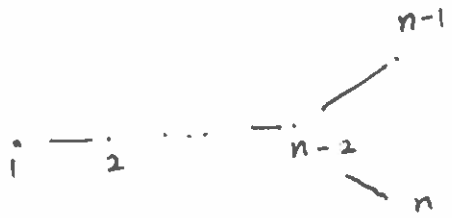

$$\begin{bmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{bmatrix} \quad - A_n \ (n \geq 1)$$



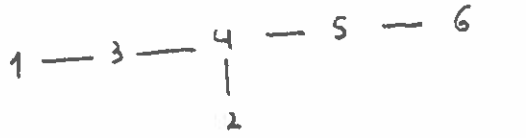

$$\begin{bmatrix} 2 & -1 & & & 0 \\ -1 & \ddots & & & \\ & \ddots & 2 & -1 & 0 \\ & & -1 & 2 & -2 \\ 0 & & 0 & -1 & 2 \end{bmatrix} \quad - B_n \ (n \geq 2)$$



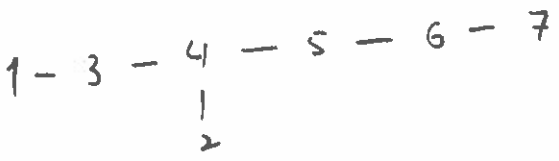

$$\begin{bmatrix} 2 & -1 & & & 0 \\ -1 & \ddots & & & \\ & \ddots & 2 & -1 & 0 \\ & & -1 & 2 & -1 \\ 0 & & 0 & -2 & 2 \end{bmatrix} \quad - C_n \ (n \geq 3)$$



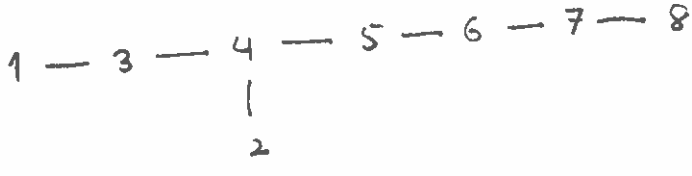

$$\begin{bmatrix} 2 & -1 & & & & 0 \\ -1 & \ddots & & & & \\ & \ddots & 2 & -1 & & \\ & & -1 & 2 & -1 & -1 \\ & & & -1 & 2 & 0 \\ 0 & & & -1 & 0 & 2 \end{bmatrix} \quad - D_n \ (n \geq 4)$$




$$\begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} \quad - E_6$$




$(E_7)$




$(E_8)$

$1 - 2 \Leftarrow 3 - 4$

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$F_4$

$1 \Lleftarrow 2$

$$\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$G_2$

## Sketch of the proof

$\mathfrak{g}$ : simple Lie algebra (over $\mathbb{C}$) ⟵ Chevalley-Serre presentation

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→ Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ exists and is unique upto action of $\underset{\text{LieAlg}}{\text{Aut}}(\mathfrak{g})$.

↓

$R \subset E \setminus \{0\}$ (irreducible) root system.

↓

→ A base $\{\alpha_i\}_{i \in I} \subset R$ exists and is unique upto the action of the Weyl group

↓

Cartan Matrix / Dynkin diagram