Lecture 22

1. Let $\mathfrak{g}$ be a finite-dimensional non-zero semisimple Lie algebra over $\mathbb{C}$.

Recall (Lecture 13, §4, page 3): $\forall\, x \in \mathfrak{g}$, $\exists!\ x_s$ and $x_N \in \mathfrak{g}$ s.t. (i) $[x_s, x_N] = 0$, (ii) $x = x_s + x_N$, and (iii) $\mathrm{ad}(x_s)$ (resp. $\mathrm{ad}(x_N)$) is semisimple (resp. nilpotent).

Corollary - The set of semisimple elements in $\mathfrak{g}$, denoted by $\mathfrak{g}^{s.s.}$, is non-~~empty~~ zero.

Proof.- If $\mathfrak{g}^{s.s.} = \{0\}$, then $\mathrm{ad}(x)$ is nilpotent $\forall\, x \in \mathfrak{g}$, which, by Engel's theorem, implies that $\mathfrak{g}$ is nilpotent. But $\mathfrak{g}$ is semisimple, hence $\{0\}$. □

2. Let $\mathfrak{a} \subset \mathfrak{g}$ be a Lie subalgebra such that

• $\mathfrak{a}$ is abelian • $\forall\, a \in \mathfrak{a}$, $\mathrm{ad}(a)$ is semisimple

(by Cor. 1 above, there is some such non-zero $\mathfrak{a}$).

Thus $\mathfrak{g} = \bigoplus_{\mu \in \mathfrak{a}^*} \mathfrak{g}_\mu$ (as representations of $\mathfrak{a}$ via adjoint)

where $\mathfrak{g}_\mu = \{ x \in \mathfrak{g} : [a, x] = \mu(a)\, x \ \ \forall\, a \in \mathfrak{a} \}$

(root space relative to $\mathfrak{a}$).

Let $K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ denote the Killing form. Recall that K is non-degenerate. (Lecture 12, §2)

Lemma. - (a) if $x \in \mathfrak{g}_\mu$ and $y \in \mathfrak{g}_\nu$ then

$$[x, y] \in \mathfrak{g}_{\mu+\nu}.$$

(b) if $x \in \mathfrak{g}_\mu$, $y \in \mathfrak{g}_\nu$ and $\mu + \nu \neq 0$, then

$$K(x, y) = 0.$$

(c) $\mathfrak{g}_0$ is reductive. ($\mathfrak{g}_0 = \{x : [a,x]=0 \ \forall a \in \mathfrak{a}\} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$; $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$ = centralizer of $\mathfrak{a}$)

Proof. - (a) For any $a \in \mathfrak{a}$, we have

$$\begin{aligned} \mathrm{ad}(a) \cdot [x, y] &= [\mathrm{ad}(a) \cdot x, y] + [x, \mathrm{ad}(a) \cdot y] \\ &= (\mu(a) + \nu(a))\,[x, y]. \end{aligned}$$

(b) As $[x, [y, z]] \in \mathfrak{g}_{\alpha+(\mu+\nu)}$ $\forall$ $z \in \mathfrak{g}_\alpha$; and $\mu+\nu \neq 0$,

$\mathrm{ad}(x) \circ \mathrm{ad}(y)$ is nilpotent, hence has trace 0.

(c) From (b) above $K\big|_{\mathfrak{g}_0 \times \mathfrak{g}_0}$ is non-degenerate. Thus, we have a representation of $\mathfrak{g}_0$ ($\mathfrak{g}_0 \xrightarrow{\mathrm{ad}} \mathfrak{gl}(\mathfrak{g})$) whose associated invariant form is non-degenerate, hence $\mathfrak{g}_0$ is reductive. (Lecture 10, §6, page 6). □

3. Cartan subalgebra

- We say $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra if

(i) $\mathfrak{h}$ is abelian.

(ii) Every $h \in \mathfrak{h}$ is semisimple.

(iii) $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.

Theorem.- Cartan subalgebras exist.

Proof - Take $\mathfrak{h} \subset \mathfrak{g}$ to be of maximum dimension among all $\{\mathfrak{a} \subset \mathfrak{g} : \mathfrak{a}$ is abelian & consists of semisimple elements$\}$

We need to prove that $[h, x] = 0, \forall\, h \in \mathfrak{h}$

$\Rightarrow x \in \mathfrak{h}$

If $x = x_s + x_N$ is the J.C decomposition of x, then $[h, x] = 0$ implies $[h, x_s] = 0$ (recall, $\mathrm{ad}(x_s)$ can be written as a polynomial $P(\mathrm{ad}(x))$, with $P(0) = 0$ - see §4 of Lecture 11 - also Lecture 13).

As x_s is semisimple and $\mathfrak{h}$ is max'l, we get $x_s \in \mathfrak{h}$.

So, without loss of generality, we may assume that x is nilpotent.

~~But in that case, for any $h \in \mathfrak{h}$~~

The argument above implies that every $x \in Z_{\mathfrak{g}}(\mathfrak{h})$ is nilpotent modulo $\mathfrak{h} \subset Z_{\mathfrak{g}}(\mathfrak{h})$. Hence $Z_{\mathfrak{g}}(\mathfrak{h})$ is contained in center of $Z_{\mathfrak{g}}(\mathfrak{h})$

a nilpotent Lie algebra, which is also reductive by Lemma 2(c) above - hence $Z_{\mathfrak{g}}(\mathfrak{h})$ is abelian.

Finally if $x \in Z_{\mathfrak{g}}(\mathfrak{h})$ is s.t. $\mathrm{ad}(x) \subset \mathfrak{g}$ is nilpotent

Then, $\forall\, y \in Z_{\mathfrak{g}}(\mathfrak{h})$; $\operatorname{ad}(x)$, $\operatorname{ad}(y)$ commute

$\Rightarrow$ $\operatorname{ad}(x) \circ \operatorname{ad}(y)$ is nilpotent operator on $\mathfrak{g}$

$\Rightarrow$ $K(x, y) = 0$.

By non-degeneracy of K restricted to $Z_{\mathfrak{g}}(\mathfrak{h})$, this implies that $x = 0$.

□

4. Root space decomposition - associated to the pair $(\mathfrak{g}, \mathfrak{h})$

($\mathfrak{g}$: non-zero semisimple, f.d., Lie algebra

$\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra)

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}_\alpha \qquad (\text{since } \mathfrak{g}_0 = Z_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}.)$$

Properties - same as the ones from Lemma 2 above :

(a) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$

(b) $\mathfrak{g}_\alpha \perp \mathfrak{g}_\beta$ if $\alpha + \beta \neq 0$

Hence K restricts to a non-degenerate pairing

$\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathbb{C}$, in particular,

(c) $(-,-) : \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}$ - denoted by $(-,-)$

(dropping K for simplicity)

Let $\nu : \mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$ be the isomorphism induced from $(-,-)$

Thus $\forall\ x \in \mathfrak{h}$, $\nu(x) : \mathfrak{h} \to \mathbb{C}$.
$y \mapsto (x,y)$

For $\gamma, \mu \in \mathfrak{h}^*$, set $\underline{(\gamma,\mu) := (\nu^{-1}(\gamma), \nu^{-1}(\mu))}$.

5. $\underline{sl_2\text{-subalgebras}}$ - Let $R = \{\alpha \in \mathfrak{h}^* \setminus \{0\} : \mathfrak{g}_\alpha \neq \{0\}\}$
(set of roots associated to the pair $(\mathfrak{g}, \mathfrak{h})$.)

For each $\alpha \in R$, choose $x_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$ s.t.
$(x_\alpha, x_{-\alpha}) = 1$.

Set $t_\alpha := [x_\alpha, x_{-\alpha}] \in \mathfrak{h}$. [Note: Under the iso. $\nu : \mathfrak{h} \to \mathfrak{h}^*$; $\nu(t_\alpha) = \alpha \neq 0$, This is because

$(t_\alpha, h) = ([x_\alpha, x_{-\alpha}], h) = -(x_\alpha, [h, x_{-\alpha}])$ (by invariance)
$= \alpha(h)\ (x_\alpha, x_{-\alpha}) = \alpha(h)$; $\forall\ h \in \mathfrak{h}$.]

$\underline{\text{Lemma}}$. $\alpha(t_\alpha) = (t_\alpha, t_\alpha) \neq 0$ (Note: $(\alpha,\alpha) = (t_\alpha, t_\alpha)$)

$\underline{\text{Proof}}$.- Let $\mathfrak{or}$ = span of $x_{\pm\alpha}, t_\alpha \subset \mathfrak{g}$
3-dim'l Lie subalgebra

If $\alpha(t_\alpha) = 0$, then $[x_\alpha, x_{-\alpha}] = t_\alpha$
$[t_\alpha, x_{\pm\alpha}] = 0$

implies $\mathfrak{or}$ is solvable. Hence, by Lie's theorem,

there is a basis of $\mathfrak{g}$ in which $\mathrm{ad}(a)$ is upper triangular, for every $a \in \mathfrak{a}$.

But then $\mathrm{ad}(t_\alpha)$ is strictly upper triangular - i.e., $(t_\alpha = [x_\alpha, x_{-\alpha}])$ nilpotent, contradicting $t_\alpha \in \mathfrak{h} \setminus \{0\}$. □

Definition. For each $\alpha \in R$, choose $e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha \in \mathfrak{g}_{-\alpha}$ so that $(e_\alpha, f_\alpha) = \frac{2}{(\alpha,\alpha)}$

$$h_\alpha = [e_\alpha, f_\alpha] = \frac{2 \cdot t_\alpha}{(\alpha,\alpha)} \in \mathfrak{h}$$

Then $\{e_\alpha, f_\alpha, h_\alpha\}$ satisfy the relations of $\mathfrak{sl}_2$.

$\mathfrak{sl}_2^{(\alpha)} := \mathrm{Span}\{e_\alpha, f_\alpha, h_\alpha\} \subset \mathfrak{g}$ ($\mathfrak{sl}_2$-subalgebra).

6. Properties of $R \subset \mathfrak{h}^* \setminus \{0\}$ (using representation theory of $\mathfrak{sl}_2$)

Prop. - (a) If $\alpha \in R$ then $\dim \mathfrak{g}_{\pm k\cdot\alpha} = \begin{cases} 1 & \text{if } k=1 \\ 0 & \text{otherwise} \end{cases}$ $(k \in \mathbb{Z}_{\geq 1})$

(b) $\forall\, \alpha, \beta \in R$, $\beta(h_\alpha) = 2\,\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$

(c) For $\alpha \in R$, define $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ be given by

$$s_\alpha(\gamma) = \gamma - \gamma(h_\alpha)\cdot\alpha$$

Then $s_\alpha(\beta) \in R$, $\forall \beta \in R$.

(d) $\mathrm{Span}(R) = \mathfrak{h}^*$.

Proof. - (d) If $\mathrm{Span}(R) \subsetneq \mathfrak{h}^*$, then $\exists\ 0 \neq x \in \mathfrak{h}$ s.t. $\alpha(x) = 0$ $\forall\ \alpha \in R$.

This means $\mathrm{ad}(x) = 0$ on each $\mathfrak{g}_\alpha$ and also on $\mathfrak{h}$,

i.e. $x \in Z(\mathfrak{g}) = \{0\}$, contradiction.

(a) Let $\alpha \in R$ and set $V_\alpha = \mathbb{C} \cdot h_\alpha \oplus \bigoplus_{k \in \mathbb{Z} \setminus \{0\}} \mathfrak{g}_{k\cdot\alpha}$.

Claim $V_\alpha \subset \mathfrak{g}$ is $\mathfrak{sl}_2^{(\alpha)}$-subrepn.

(i.e. $\mathrm{ad}(x) : V_\alpha \to V_\alpha$ $\forall\ x \in \{e_\alpha, f_\alpha, h_\alpha\}$)

Pf. of the claim - The only non-trivial verification is

$$\mathrm{ad}(e_\alpha) : \mathfrak{g}_{-\alpha} \to \mathbb{C} \cdot h_\alpha$$

(similarly, $\mathrm{ad}(f_\alpha) : \mathfrak{g}_\alpha \to \mathbb{C} \cdot h_\alpha$).

But we have proved, more generally (see page 5 above)

that $\forall\ x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$, $[x, y] = (x, y) \cdot t_\alpha \in \mathbb{C} \cdot h_\alpha$ □

Note: $P(V_\alpha) \subset 2 \cdot \mathbb{Z}$ all weights are even

$\dim V_\alpha[0] = 1$ $(V_\alpha[0] = \mathbb{C} \cdot h_\alpha)$

$\Rightarrow V_\alpha$ is irreducible, i.e. $\exists\ \ell \geq 1$ s.t.

$V_\alpha = \mathfrak{g}_{l\alpha} \oplus \dots \oplus \mathfrak{g}_{-l\alpha}$ (weight-decomposition)

$2l, 2l-2, \dots, -2l+2, -2l$ weights ($\mathrm{ad}(h_\alpha)$-eigenvalues)

and $\dim \mathfrak{g}_{r\alpha} = 1 \quad \forall\ -l \le r \le l$.

But then $\mathfrak{g}_\alpha = \mathbb{C} \cdot e_\alpha$ and $\mathrm{ad}(e_\alpha) \cdot e_\alpha = 0$

↑ highest-weight vector of weight 2

$\Rightarrow l = 1$ and we are done.

(b) $\mathfrak{sl}_2^{(\alpha)} \circlearrowright \mathfrak{g}$ f.d. repn. $\Rightarrow$ eigenvalues of $\mathrm{ad}(h_\alpha)$ are integers

$\Rightarrow \forall \beta \in R, \ \beta(h_\alpha) \in \mathbb{Z}$.

(c) Let $\alpha \in R$, $\beta \in R \setminus \{\pm\alpha\}$. Assume $\beta(h_\alpha) \ge 0$. Then,

by $\mathfrak{sl}_2$-repn. theory, $\mathrm{ad}(f_\alpha)^{\beta(h_\alpha)} \cdot \mathfrak{g}_\beta \neq 0$

$$\mathrm{ad}(f_\alpha)^{\beta(h_\alpha)} : \mathfrak{g}_\beta \longrightarrow \mathfrak{g}_{\underbrace{\beta - \beta(h_\alpha)\cdot\alpha}_{= S_\alpha(\beta)}}.$$

□