Lecture 23

1. Recall: $\mathfrak{g}$ is a finite-dimensional, semisimple Lie algebra / $\mathbb{C}$.

(a) Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. (Lecture 22, Thm. §3).

Then, we obtain the root space decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \quad ; \quad \text{where, for } \mu \in \mathfrak{h}^*, \text{ we define}$$

$$\mathfrak{g}_\mu := \{ x \in \mathfrak{g} : [h, x] = \mu(h)\, x, \ \forall\, h \in \mathfrak{h} \}$$

and $R = \{ \alpha \in \mathfrak{h}^* - \{0\} : \mathfrak{g}_\alpha \neq \{0\} \}$.

(set of roots associated to the pair $(\mathfrak{g}, \mathfrak{h})$)

(b) $K = K_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ Killing form on $\mathfrak{g}$ - restricts to non-degenerate, (symmetric) invariant form

$$K : \mathfrak{g}_\gamma \times \mathfrak{g}_{-\gamma} \to \mathbb{C}, \quad \forall\, \gamma \in R \cup \{0\}$$

(recall: $\mathfrak{g}_0 = Z_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.)

Let $\nu : \mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$ be the resulting (linear) isomorphism

$$\left[\begin{array}{l} \nu(x)(y) = (x, y) \\ \forall\ x, y \in \mathfrak{h}. \end{array} \right]$$

$(-,-)$ will denote $K|_{\mathfrak{h} \times \mathfrak{h}}$; or its transport to $\mathfrak{h}^*$ via ν.

$$t_\alpha := \nu^{-1}(\alpha) \quad \text{and} \quad h_\alpha := \frac{2\, t_\alpha}{(\alpha, \alpha)}$$

Important note For $x, y \in \mathfrak{h}$,

$$(x, y) = \sum_{\beta \in R} (\dim \mathfrak{g}_\beta)\, \beta(x)\, \beta(y) \qquad - (*)$$

In the previous lecture, we proved:

(i) $\dim \mathfrak{g}_\alpha = 1 \quad \forall \alpha \in R$

(ii) R spans $\mathfrak{h}^*$. If $\alpha, c\alpha \in R$, then $c = \pm 1$.

(iii) $\forall \alpha, \beta \in R$, $\beta(h_\alpha) = \dfrac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer. $c \in \mathbb{Z}$ (†)

(iv) $\forall \alpha \in R$, $s_\alpha(R) \subset R$, where $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ is given by

$$s_\alpha(\gamma) = \gamma - \gamma(h_\alpha)\alpha$$

2. Real forms and positive-definite-ness.

Let $\mathfrak{h}_{\mathbb{R}}$ = $\mathbb{R}$-span of $\{h_\alpha\}_{\alpha \in R}$

Cor. Killing form restricts to a positive-definite form on $\mathfrak{h}_{\mathbb{R}}$

Pf. Note $\beta(h_\alpha) \in \mathbb{R}$ $\forall \alpha, \beta \in R$. Thus, $\forall\, t \in \mathfrak{h}_{\mathbb{R}}$,

$\beta(t) \in \mathbb{R}$. Hence, by (*) on the previous page, we have

$$(t,t) = \sum_{\beta \in R} \beta(t)^2 \geq 0.$$ Note - if $\beta(t) = 0$ $\forall \beta \in R$, then $t \in Z(\mathfrak{g}) = \{0\}$

So, $(t,t) = 0 \iff t = 0$. □

$\mathfrak{h}^*_{\mathbb{R}}$ is then identified with the real span of $\{\alpha : \alpha \in R\}$.

Remark. As $(h_\alpha, h_\alpha) = \dfrac{4(\alpha,\alpha)}{(\alpha,\alpha)^2} = \dfrac{4}{(\alpha,\alpha)} \in \mathbb{R}$,

we conclude that $(\alpha,\alpha) \in \mathbb{R}$.

(†) - note, by (iii) $\alpha \in R$, $c\alpha \in R \Rightarrow (c\alpha)(h_\alpha) = 2c \in \mathbb{Z}$, $c \in \mathbb{R}$

3. Root system. Let E be a real vector space together with a positive definite (symmetric, bilinear) form $(-,-) : E \times E \to \mathbb{R}$; $v : E \to E^*$ corresponding iso. and $(-,-) : E^* \times E^* \to \mathbb{R}$ transported, so v is an isometry.

A finite set $R \subset E^* \setminus \{0\}$ is called a root system if the following axioms hold:

(i) R spans E^* and $\alpha, c\alpha \in R \Rightarrow c = \pm 1$.

(ii) $\forall \alpha, \beta \in R$, $\dfrac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$

In the notations of the previous paragraphs, $t_\alpha := v^{-1}(\alpha)$

$$h_\alpha := \frac{2t_\alpha}{(\alpha,\alpha)}$$

$$\beta(h_\alpha) = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$$

(iii) For $\alpha \in R$, $S_\alpha(R) \subset R$, where

$$S_\alpha : E^* \to E^*$$
$$\gamma \mapsto \gamma - \gamma(h_\alpha)\alpha$$

[Similarly the same symbol denotes $S_\alpha : E \to E$, $x \mapsto x - \alpha(x) h_\alpha$]

(Note: S_α is an isometry:

$$(S_\alpha(\gamma_1), S_\alpha(\gamma_2)) = (\gamma_1 - \gamma_1(h_\alpha)\alpha, \gamma_2 - \gamma_2(h_\alpha)\alpha)$$

$$= (\gamma_1, \gamma_2) - \gamma_2(h_\alpha)\,(\gamma_1, \alpha) - \gamma_1(h_\alpha)\,(\gamma_2, \alpha) + \gamma_1(h_\alpha)\gamma_2(h_\alpha)\,(\alpha, \alpha)$$

Using $(\gamma_1, \alpha) = \frac{(\alpha,\alpha)}{2}\,\gamma_1(h_\alpha)$, the last three terms cancel and we get $(S_\alpha(\gamma_1), S_\alpha(\gamma_2)) = (\gamma_1, \gamma_2)$.)

Remark. - We say that a root system is irreducible if E^* cannot be written as a direct sum of orthogonal subspaces $E^* = E_1^* \oplus E_2^*$ $\left[(\gamma_1, \gamma_2) = 0 \;\; \forall\, \gamma_1 \in E_1^*,\ \gamma_2 \in E_2^* \right]$

so that $R = (R \cap E_1^*) \sqcup (R \cap E_2^*)$.

Ex. of simple $\Rightarrow$ the root system associated to (g, f) is irreducible.

4. Base of a root system. - A subset $\Delta = \{\alpha_i\}_{i \in I} \subset R$ is called a base of R if

(i) Δ is a basis (of E^*)

(ii) $R = R_+ \sqcup R_-$ where $R_\pm = R \cap \left(\pm \sum_{i \in I} \mathbb{Z}_{\geq 0}\, \alpha_i \right)$ $\left[\begin{array}{c} \text{Sign coherence} \\ \text{or} \\ \text{polarization} \end{array} \right]$

Theorem. - Let $R \subset E^* \setminus \{0\}$ be a root system, as above.
Then R has a base.

Proof. - For $\alpha \in E^* \setminus \{0\}$ let $H_\alpha = \mathrm{Ker}(\alpha) \subset E$
hyperplane defined by α

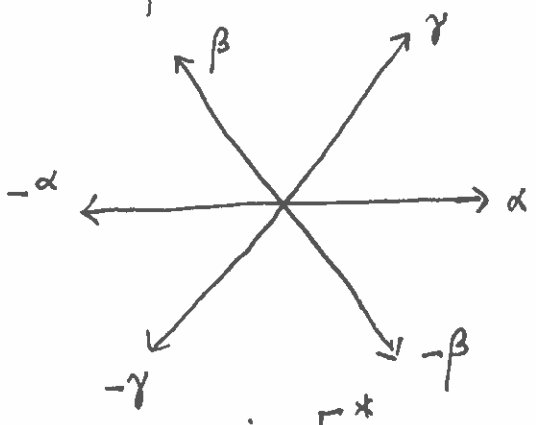
Note $H_\alpha = H_\beta \iff \alpha$ and β are proportional.

Thus from $R \subset E^* \setminus \{0\}$, we get a hyperplane arrangement $\{H_{-\alpha} = H_\alpha \subset E\}_{\alpha \in R}$. The complement $E^\circ = E \setminus \bigcup_{\alpha \in R} H_\alpha$ is a disconnected space.

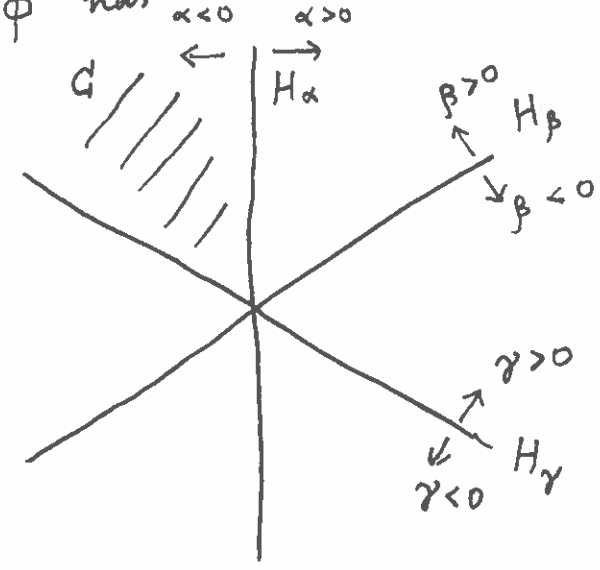
The connected components of E° are called chambers.
If $C \subset E^\circ$ is a chamber, the set of walls of C are linear forms $\alpha \in R$ so that

$$\begin{cases} \alpha(x) > 0 & \forall\ x \in C \\ H_\alpha \cap \overline{C} \neq \phi & \text{has codimension 1} \end{cases}$$

Example A_2 root system



in E^*



in E

If C is the shaded connected component above then $\text{Walls}(C) = \{-\alpha, \gamma\}$.

——— end of example —

Let $C \subset E^\circ$ be a fixed chamber - we will call it the fundamental (or the Weyl) chamber. Accordingly - we obtain a polarization of $R = R_+ \sqcup R_-$, where,

$$R_+ = \{\alpha \in R : \alpha > 0 \text{ on } C\}$$
$$R_- = \{\alpha \in R : \alpha < 0 \text{ on } C\}$$

Since C is connected, no H_α intersects C - hence every root is either in R_+ or in R_-. Moreover, in our situation, $\forall \alpha \in R$, $-\alpha = s_\alpha(\alpha) \in R$ implies that $R_- = -R_+$.

[Lemma.- Let V be a real vector space (finite-dim'l). $\Lambda \subset V^* \setminus \{0\}$ s.t. distinct elements of Λ are not proportional.

$$\Sigma = \text{Cone}(\Lambda) := \left\{ \sum_\lambda c_\lambda \cdot \lambda : c_\lambda \in \mathbb{R}_{\geq 0} \right\} \subset V^*$$

$$I_>(\Lambda) := \{x \in V : \lambda(x) > 0 \ \forall \lambda \in \Lambda\} \subset V$$

Then $\text{Walls}(I_>(\Lambda)) = \text{Rays of } \Sigma$, where $\lambda \in \Lambda$ is a ray of Σ if $\text{Cone}(\Lambda \setminus \{\lambda\}) \subsetneq \text{Cone}(\Lambda)$; or equivalently $\lambda \notin \text{Cone}(\Lambda \setminus \{\lambda\})$.]

Proof of this lemma is omitted here - will be given in the next lecture. We return to the proof of our theorem. So, let $\Delta = \{\alpha_i\}_{i \in I} \subset R_+$ be the set of walls of C. By the lemma above

$$R_+ \subset \text{Cone}(\Delta) = \left\{ \sum_{i \in I} c_i \alpha_i : c_i \in \mathbb{R}_{\geq 0} \right\}$$

As R_+ spans E^*, we get that Δ spans E^*.

Claim. - $\forall\ i \neq j, \quad (\alpha_i, \alpha_j) \leq 0$.

(Given the claim, linear independence of $\{\alpha_i\}_{i \in I}$ follows:

let $\sum_{i \in I} c_i \alpha_i = 0$ be a non-trivial dependence relation.

$I_1 := \{i : c_i > 0\}$, $I_2 := \{j : c_j < 0\}$ are disjoint

and $\gamma = \sum_{i \in I_1} c_i \alpha_i = \sum_{j \in I_2} d_j \alpha_j \qquad (d_j = -c_j > 0)$.

$$\Rightarrow (\gamma, \gamma) = \sum_{\substack{i \in I_1 \\ j \in I_2}} c_i d_j \underbrace{(\alpha_i, \alpha_j)}_{\leq 0} \leq 0 \quad \text{contradiction.})$$

Proof of the claim. - if $\exists\ i \neq j$ s.t. $(\alpha_i, \alpha_j) > 0$ then

$$\gamma = s_{\alpha_i}(\alpha_j) = \alpha_j - c\alpha_i \qquad (c \in \mathbb{Z}_{>0})$$

is a root. If $\gamma \in R_+$ then $\alpha_j = c\alpha_i + \gamma$

$\Rightarrow \alpha_j \in \text{Cone}(R_+ \setminus \{\alpha_j\})$ contradicting that it is a ray.

Similarly if $\gamma \in R_-$, then $c\alpha_i = \alpha_j + (-\gamma)$ and we get a contradiction.

It remains to prove that $R_+ \subset \sum_{i \in I} \mathbb{Z}_{\geq 0}\, \alpha_i$.

For this, note that if $\alpha \in R_+ \setminus \{\alpha_i\}_{i \in I}$, then there exists some $i \in I$ s.t. $(\alpha, \alpha_i) > 0$.

[Pf. 1 - If $(\alpha, \alpha_i) \leq 0 \ \forall i$, and $\alpha = \sum_{i \in I} c_i \alpha_i$, $c_i \in \mathbb{R}_{\geq 0}$ then $(\alpha, \alpha) = \sum c_i (\alpha, \alpha_i) \leq 0$ - contradiction.

Pf 2. If $(\alpha, \alpha_i) \leq 0 \ \forall i$, then (by the proof of the claim) $\{\alpha_i\}_{i \in I} \cup \{\alpha\}$ is linearly independent. But we already know $\{\alpha_i\}_{i \in I}$ is a basis.]

For $\alpha = \sum_{j \in I} c_j \alpha_j \in R_+ \subset \mathrm{Cone}(\{\alpha_i\})$, let $\boxed{\mathrm{ht}(\alpha) := \sum_{j \in I} c_j}$ (height of α).

If $R_+ \not\subset \sum_{j \in I} \mathbb{Z}_{\geq 0}\, \alpha_j$, let $\alpha \in R_+ \setminus \sum_{j \in I} \mathbb{Z}_{\geq 0}\, \alpha_j$ be of smallest height. Then

$$s_{\alpha_i}(\alpha) = \alpha - \alpha(h_i)\alpha_i$$

$$= \sum_{j \neq i} c_j \alpha_j + (c_i - \alpha(h_i))\alpha_i$$

is still in R_+ ($\alpha \neq \alpha_i$, so some $c_j > 0$ for $j \neq i$) and has smaller height. Contradiction. □