Lecture 24.

1. Lemma from Lecture 23, page 6 -

Set up: V is a finite-dim'l $\mathbb{R}$-vector space.

$\Lambda \subset V^* \setminus \{0\}$ a finite set s.t. $\lambda \neq \mu$ in Λ are not proportional

$$\sum = \mathrm{Cone}(\Lambda) := \left\{ \sum_{\lambda \in \Lambda} c_\lambda \lambda \;:\; c_\lambda \in \mathbb{R}_{\geq 0} \right\} \subset V^*$$

$$C = I_>(\Lambda) := \{ v \in V \;:\; \lambda(v) > 0 \;\; \forall\, \lambda \in \Lambda \} \subset V$$

$$\mathrm{Rays}(\Sigma) := \{ \lambda \in \Lambda \;:\; \mathrm{Cone}(\Lambda \setminus \{\lambda\}) \subsetneq \mathrm{Cone}(\Lambda) \text{ or } \lambda \notin \mathrm{Cone}(\Lambda \setminus \{\lambda\}) \}$$

$$\mathrm{Walls}(C) := \{ \mu \in \Lambda \;:\; \exists\, v \in \bar{V} \text{ s.t. } \lambda(v) \geq 0 \;\; \forall\, \lambda \in \Lambda \setminus \{\mu\},\; \mu(v) < 0 \}$$

Remark.- Thus, $\mu \in \Lambda$ is a wall of C if $\mu > 0$ is not a consequence of $\lambda > 0 \;\; \forall\, \lambda \in \Lambda \setminus \{\mu\}$. That is,

~~Thus~~ ~~$I_>(\Lambda \setminus \{\mu\})$ and $I_>($~~

$I_>(\Lambda \setminus \{\mu\})$ meets $\mathrm{Ker}(\mu) \Rightarrow \bar{C} \cap \mathrm{Ker}(\mu)$ has codim 1.

Lemma.- $\text{Rays}(\Sigma) = \text{Walls}(C)$

Proof.- If $\alpha \notin \text{Rays}(\Sigma)$ i.e. $\alpha \in \text{Cone}(\Lambda \setminus \{\alpha\})$ can be written as $\alpha = \sum_{\beta \in \Lambda \setminus \{\alpha\}} c_\beta \beta$. Then, $\forall u \in V$ s.t. $(c_\beta \in \mathbb{R}_{\geq 0})$

$\lambda(u) \geq 0$, $\lambda \in \Lambda \setminus \{\alpha\}$, we get $\alpha(u) \geq 0$, implying that α is not a wall of C.

Conversely, if $\alpha \in \text{Rays}(\Sigma)$, then $\alpha \notin \text{Cone}(\Lambda \setminus \{\alpha\})$.

By separating hyperplane lemma, $\exists\ v \in V$ s.t.

$$\lambda(v) \geq 0 \quad \forall\ \lambda \in \text{Cone}(\Lambda \setminus \{\alpha\})$$
$$\alpha(v) < 0$$

□

2. Recap.- E is a f.d. Euclidean space

$$\nu : E \xrightarrow{\sim} E^* \qquad t_\alpha := \nu^{-1}(\alpha)$$
$$h_\alpha := \frac{2 t_\alpha}{(t_\alpha, t_\alpha)}$$

$R \subset E^* \setminus \{0\}$ root system.

$$E^\circ = E \setminus \bigcup_{\alpha \in R} H_\alpha \qquad (H_\alpha = \text{Ker}(\alpha) \subset E \text{ hyperplane})$$

$\forall \alpha \in R$, $S_\alpha : E \to E$ or $S_\alpha : E^* \to E^*$

$$x \mapsto x - \alpha(x) h_\alpha \qquad \gamma \mapsto \gamma - \gamma(h_\alpha)\alpha$$

W = group generated by $\{S_\alpha\}_{\alpha \in R}$. [Weyl group]

Properties. - (easy to verify - Ex.)

(i) $\nu : E \xrightarrow{\sim} E^*$ is W-intertwiner

(ii) $w(\mathrm{Ker}(\alpha)) = \mathrm{Ker}(\bar{w}^t(\alpha))$. Hence W acts on E°

(iii) $W \subset$ Permutations of R, hence $|W| < \infty$.

3. Simple roots and simple reflections.

Fix a chamber $C \subset E^\circ$, let $R = R_+ \sqcup R_-$ be the corresponding polarization and $\{\alpha_i\}_{i\in I} \subset R_+$ the base.

α_i's are called simple roots

$s_i = s_{\alpha_i}$ are called simple reflections.

Proposition. - W is generated by $\{s_i\}_{i\in I}$. Moreover $R = \bigcup_{i\in I} W\cdot\alpha_i$.

Proof. - Let $W' \subset W$ be the subgroup generated by $\{s_i\}_{i\in I}$

Again, assume $\exists\, \alpha \in R_+ \setminus \bigcup_{i\in I} W'\cdot\alpha_i$ and choose one of min. height. As $\alpha \notin \{\alpha_i\}_{i\in I}$, we can find i s.t. $\alpha(h_i) > 0$. Then (see Lecture 23 - page 8)

$$s_i(\alpha) \in R_+ \setminus \bigcup_{i\in I} W'\cdot\alpha_i$$ has smaller height

- contradiction. Thus $R_+ \subseteq \bigcup_{i\in I} W'\cdot\alpha_i$.

If $\alpha \in R_+$, $\alpha = w(\alpha_i)$ for some $w \in W'$, $i \in I$.,

then $S_\alpha = w s_i w^{-1} \in W'$.

Since W is generated by $\{S_\alpha\}_{\alpha \in R_+}$, we get $W' = W$.

Also $-\alpha = w(-\alpha_i) = w s_i(\alpha_i) \in \bigcup_{i \in I} W \cdot \alpha_i$.

So. $R = \bigcup_{i \in I} W \cdot \alpha_i$ as claimed. $\square$

4. Rank 2 classification. - $(E, R \subset E^* - \{0\}, \{\alpha_i\}_{i \in I}, W)$ - as above

$A = (a_{ij})_{i,j \in I}$ defined by $a_{ij} = \alpha_j(h_i) = \dfrac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$

is called the Cartan matrix of the root system R.

$|I| = \dim(E)$ - rank of R.

When $|I| = 2$, A has the form $\begin{bmatrix} 2 & -a \\ -b & 2 \end{bmatrix}$, $a, b \in \mathbb{Z}_{\geq 0}$

$a = 0 \Rightarrow b = 0$.

Moreover, $\det(A) = 4 - ab > 0$. Thus (up to permuting the labels of rows & columns of A), we have the following possibilities:

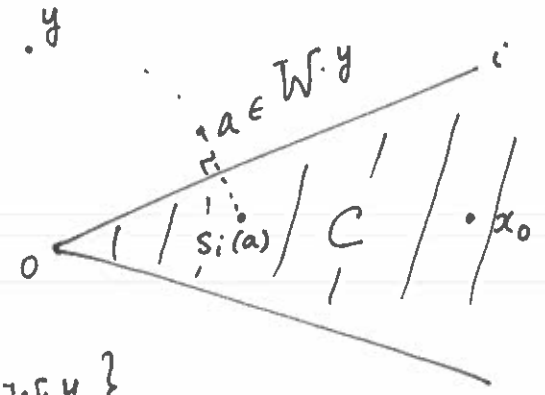
$a = b = 0$	$A_1 \times A_1$
$a = 1$ $b = 1$	A_2
$a = 2$ $b = 1$	B_2
$a = 3$ $b = 1$	G_2

5. Choice of the chamber -

Thm.- W action on the set of chambers is transitive.

Proof.- We will prove that for any $y \in E$, we can find $w \in W$ such that $w(y) \in \overline{C}$.

Pick $x_0 \in C$ and let $a \in W \cdot y$ be such that



$$\text{distance}(x_0, a) = \text{MIN}\{ d(x_0, b) : b \in W \cdot y \}$$

Claim.- $a \in \overline{C}$ i.e. $\alpha_i(a) \geq 0 \quad \forall\, i \in I$.

Otherwise, $\exists\, i$ s.t. $\alpha_i(a) < 0$. Then,

$$|x_0 - s_i(a)|^2 - |x_0 - a|^2 = |x_0 - a + \alpha_i(a) h_i|^2 - |x_0 - a|^2$$

$$= \alpha_i(a)^2 (h_i, h_i) + 2\alpha_i(a)(x_0 - a, h_i)$$

$$= 2\alpha_i(a)(x_0, h_i) + \alpha_i(a)^2 (h_i, h_i) - 2\alpha_i(a)(a, h_i)$$

$$\left[(a, h_i) = \frac{2\alpha_i(a)}{(\alpha_i, \alpha_i)} \quad \text{and} \quad (h_i, h_i) = \frac{4}{(\alpha_i, \alpha_i)} \right]$$

$$= \frac{4\alpha_i(a)\alpha_i(x_0)}{(\alpha_i, \alpha_i)} < 0 \quad \text{- i.e. } d(x_0, s_i(a)) < d(x_0, a)$$

contradiction. □