Lecture 25

1. Cartan matrices - (finite type) - Let I be a finite set and $A \in M_{I\times I}(\mathbb{Z})$. We say A is a (finite-type) Cartan matrix if

(i) $a_{ii} = 2 \quad \forall\ i \in I$

(ii) $a_{ij} \in \mathbb{Z}_{\leq 0} \quad \forall\ i \neq j$ in I.

(iii) A is symmetrizable - i.e., $\exists\ d_i \in \mathbb{R}_{>0}$ $(i \in I)$ so that

D·A is symmetric ($d_i a_{ij} = d_j a_{ji} \quad \forall\ i,j \in I$).

[Note d_i's can be chosen to be in $\mathbb{Z}_{>0}$.]

(iv) A is positive-definite. (finite type condition)

Last time we showed that (via rank 2 classification) that, if $i \neq j$ in I, then the 2×2 submatrix $\begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix}$ is one of the following: (up to $i \leftrightarrow j$)

$A_1 \times A_1$

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

A_2

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

B_2

$$\begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}$$

G_2

$$\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

Ex.- Verify that these possibilities do arise from root systems

2 Dynkin diagram of A is a graph Γ_A on the vertex set I, and 3 type of edges

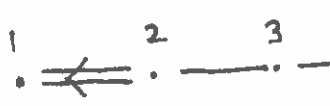
$i \text{ —— } j$ if $a_{ij} = a_{ji} = -1$ (A_2)

$i \Longleftarrow j$ if $a_{ij} = -2$, $a_{ji} = -1$ (B_2)

$i \Lleftarrow j$ if $a_{ij} = -3$, $a_{ji} = -1$ (G_2)

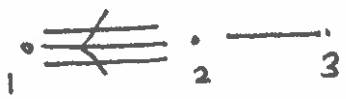
i & j not connected if $a_{ij} = a_{ji} = 0$.

(Note - by symmetrizability, if $a_{ij} = 0$ then $a_{ji} = 0$.)

Ex. 

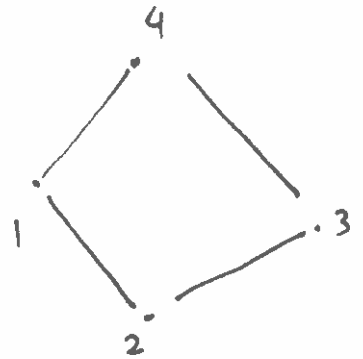
corr. to $\begin{bmatrix} 2 & -2 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

Non-examples. -



$\begin{bmatrix} 2 & -3 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ is not positive-definite

(the vector $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \mapsto 0$.)



$\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$ sum of columns = 0.

Remarks.- (1) Dynkin diagrams encode matrices over $\mathbb{Z}$ whose every 2×2 principal submatrix is of finite-type.

(2) In V. Kac - Infinite-dim'l Lie algebras, generalized (symmetrizable) Cartan matrices are considered. For instance - affine type is when we relax the positive-definiteness to positive semidefinite. In rank 2, we have two more possibilities $\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ $\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$

$A_1^{(1)}$ $A_2^{(2)}$

In Borcherd's theory - the condition on diagonal entries $= 2$ is dropped (also for "Cartan matrices associated to quivers")

if $\mathcal{G}$ is an unoriented graph (loops, multi-edges allowed) on vertex set I $\rightsquigarrow$ $A = (a_{ij})_{i,j\in I}$

$$a_{ij} = \begin{cases} 2 - 2g_i & (i=j), \\ -\#\{i-j\} & (i\neq j). \end{cases}$$

if $g_i = \#\{i \circlearrowright\}$ loops at i

3. Classification theorem.- Let $A = (a_{ij})_{i,j\in I}$ be a Cartan matrix (finite-type) as in §1 above and Γ_A be its Dynkin diagram.

We say <u>A is indecomposable</u> if Γ_A is connected - i.e.

$\forall\, i,j \in I$, $\exists\, i_1, \dots, i_r$ s.t. $a_{ii_1} a_{i_1 i_2} \cdots a_{i_{r-1} i_r} a_{i_r j} \neq 0$.

This is equivalent to requiring that I cannot be written as a non-trivial disjoint union $I = I_1 \sqcup I_2$ s.t. $a_{i_1 i_2} = 0$

$\forall\, i_1 \in I_1, i_2 \in I_2$.

(and only one)

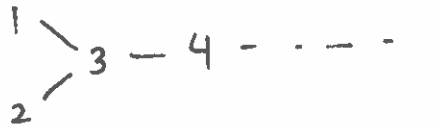
<u>Theorem.-</u> Γ_A is one of the following:

A_n $(n \geq 1)$ $\quad 1 - 2 - \cdots - n$

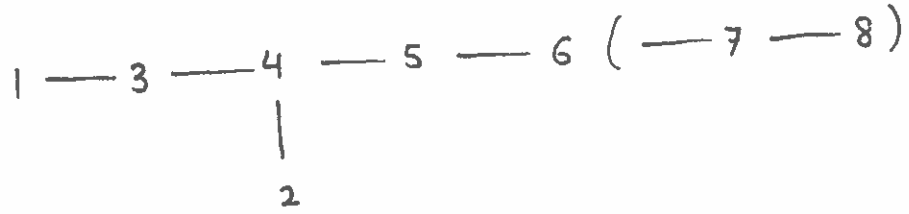
B_n $(n \geq 2)$ $\quad 1 - 2 - \cdots - n-1 \Rightarrow n$

C_n $(n \geq 3)$ $\quad 1 - 2 - \cdots - n-1 \Leftarrow n$

D_n $(n \geq 4)$



$E_{6,7,8}$



F_4 $\quad 1 - 2 \Rightarrow 3 - 4$

G_2 $\quad 1 \Rrightarrow 2$

<u>Remark.-</u> Classification of affine type Cartan matrices is given in Kac's book, Ch. 5.

4. Proof of the classification theorem - The proof is by exclusion. Note: If $\exists\ \underline{u} \in \mathbb{R}^I_{\geq 0}$, $\underline{u} \neq 0$ s.t. $A\underline{u} \leq 0$, then A cannot be a finite-type Cartan matrix.

Also, if $\exists\ J \subset I$ s.t. $(a_{j,j'})_{j,j' \in J}$ is not a Cartan matrix then A is not a Cartan matrix.

Finally - in order to exhibit $\underline{u} \in \mathbb{R}^I_{\geq 0} \setminus \{0\}$ s.t. $A\underline{u} \leq 0$, we may replace A by $\widetilde{A} \geq A$ (i.e. $\widetilde{A} - A$ has entries from $\mathbb{R}_{\geq 0}$).

In our arguments below, this will allow us to replace an edge of Γ_A by "larger edge" in the ordering



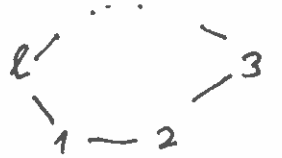
e.g.



not Γ_A for a finite-type Cartan matrix A

also not a Dynkin diagram.

(i) Γ_A has no cycles. - By the foregoing, it is enough to observe that



is not a Dynkin diagram

(ii) If Γ_A contains $\cdot\!\Leftarrow\!\cdot$ then $\Gamma_A = \cdot\!\Leftarrow\!\cdot$

Again – by connectedness of Γ_A – if it contains $\cdot\!\Leftarrow\!\cdot$, and $|I| \geq 3$, then there must be another vertex joined to one of vertices of $\cdot\!\Leftarrow\!\cdot$. Thus it is enough to show that

$\cdot\!-\!\cdot\!\Leftarrow\!\cdot$
$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\cdot\!\Leftarrow\!\cdot\!-\!\cdot$
$$\begin{bmatrix} 2 & -3 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

are not Dynkin diagrams

(iii) Γ_A can have at most one $\cdot\!\Leftarrow\!\cdot$.

If there are more than one such edge, they must be joined. Thus it suffices to exclude

$\cdot\!\Leftarrow\!\cdot\cdots\cdot\!\Leftarrow\!\cdot$
$$\begin{bmatrix} 2 & -2 & & & 0 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -2 \\ 0 & & & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ \vdots \\ 2 \\ 1 \end{bmatrix} = 0$$

$\cdot\!\Rightarrow\!\cdot\cdots\cdot\!\Leftarrow\!\cdot$

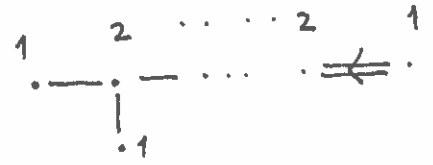
$\cdot\!\Leftarrow\!\cdot\cdots\cdot\!\Rightarrow\!\cdot$
$$\begin{bmatrix} 2 & -1 & & & \\ -2 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2 & -2 \\ & & & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ \vdots \\ 2 \\ 1 \end{bmatrix} = 0$$

$\cdot\!\Rrightarrow\!\cdot$

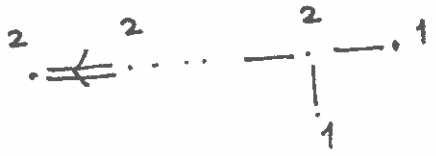
The third one is transpose of the second.

(iv) Assume Γ_A has exactly one $\cdot \Leftarrow \cdot$:

More exclusions. - no vertex of degree ≥ 3.



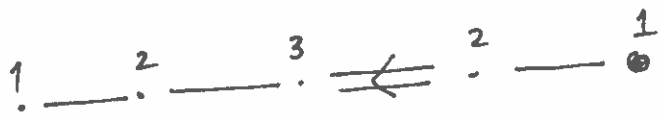
(components of the vector that goes to 0)



If both vertices of $\cdot \Leftarrow \cdot$ are connected to other vertices of Γ_A

then Γ_A =  (F_4)

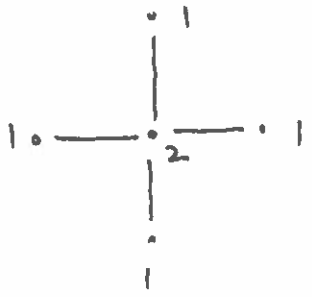
Again we can exclude 



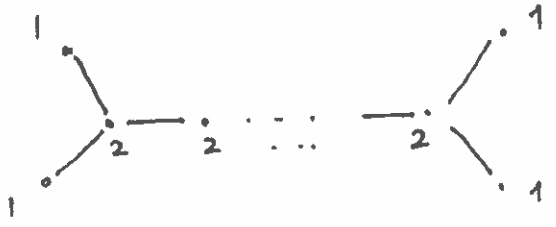
Hence, if Γ_A has a double edge $\cdot \Leftarrow \cdot$ then it is

B_n, C_n or F_4.

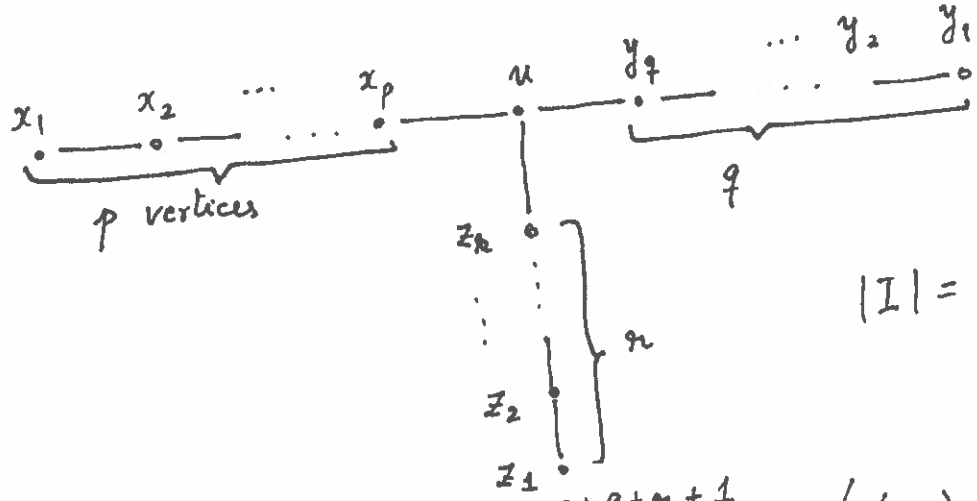
[From now on, we assumed that all edges are simple.
Thus, A is symmetric.]

(V) No vertex of deg ≥ 4: 

(vi) At most one vertex of deg = 3.



Last step.- At this point, we know that if all the edges of Γ_A are simple, then Γ_A has the following form – for some $p, q, r \in \mathbb{Z}_{\geq 0}$, say $p \leq q \leq r$.



$|I| = p+q+r+1$.

If we write a vector in $\mathbb{R}^{p+q+r+1}$ as $v = \left((x_a)_{1\leq a\leq p}, (y_b)_{1\leq b\leq q}, (z_c)_{1\leq c\leq r}, u \right)$ (as in the picture)

then under the symmetric bilinear form induced by A,

$$\frac{1}{2}(v,v) = \sum_{a=1}^{p} x_a^2 + \sum_{b=1}^{q} y_b^2 + \sum_{c=1}^{r} z_c^2 + u^2 \quad \text{[Ex: Check this.]}$$

$$- \sum_{a=1}^{p-1} x_a x_{a+1} - \sum_{b=1}^{q-1} y_b y_{b+1} - \sum_{c=1}^{r-1} z_c z_{c+1} - u(x_p + y_q + z_r)$$

Now, successive completion of squares gives ($x_{p+1} = u$ here)

$$x_1^2 + x_2^2 + \cdots + x_p^2 - x_1x_2 - x_2x_3 \cdots - x_px_{p+1} = \left(x_1 - \tfrac{1}{2}x_2\right)^2 + \tfrac{3}{4}x_2^2 + x_3^2 + \cdots + x_p^2 - x_2x_3 \cdots - x_px_{p+1}$$

$$= \left(x_1 - \tfrac{1}{2}x_2\right)^2 + \tfrac{3}{4}\left(x_2 - \tfrac{2}{3}x_3\right)^2 + \tfrac{4}{6}x_3^2 + x_4^2 + \cdots + x_p^2 - x_3x_4 \cdots - x_px_{p+1} = \cdots$$

$$= \sum_{i=1}^{p} \frac{i+1}{2i}\left(x_i - \frac{i}{i+1}x_{i+1}\right)^2 - \frac{p}{2(p+1)}x_{p+1}^2$$

$$\Rightarrow \frac{1}{2}(v,v) = \text{Sum of squares } \sum_{i=1}^{p} \frac{i+1}{2i}\left(x_i - \frac{i}{i+1}x_{i+1}\right)^2 + y\text{-term} + z\text{-term}$$

$$[x_{p+1} = y_{q+1} = z_{r+1} = u]$$

$$+ u^2\left(1 - \frac{p}{2(p+1)} - \frac{q}{2(q+1)} - \frac{r}{2(r+1)}\right) \geq 0 \quad \forall\, \underline{x}, \underline{y}, \underline{z}$$

$0 \leq p \leq q \leq r$

$p, q, r \in \mathbb{Z}_{\geq 0}$ s.t. :

$$\frac{p}{2(p+1)} + \frac{q}{2(q+1)} + \frac{r}{2(r+1)} < 1$$

$p = 0$, $q \leq r$ arbitrary, or (A)

$p = 1$, $q = 1$, $r \geq 1$ arbitrary (D)

$p = 1$, $q = 2$, $r = 2, 3, 4$ E_6, E_7, E_8.

□