# Lecture 26

1. Recall the set up: • $E$ is a finite-dim'l real Euclidean space

• $R \subset E^* \setminus \{0\}$ (finite) root system.

• $W = \langle s_\alpha : \alpha \in R \rangle \subset GL(E)$ or $GL(E^*)$ Weyl group

• $\nu : E \xrightarrow{\sim} E^*$ induced from $(\cdot,\cdot) : E \times E \to \mathbb{R}$

Note - For $x, y \in E$ ; $\gamma \in E^*$ and Weyl group element $w \in W$

$(w(x), w(y)) = (x, y)$ (also page 3 of Lecture 24)

$\nu(w(x)) = w(\nu(x))$ - see page 4 of Lecture 23 - proof that $s_\alpha \in GL(E)$

$(w(\gamma))(w(x)) = \gamma(x)$ is an isometry.

• Let $\Delta = \{\alpha_i\}_{i \in I} \subset R$ be a base of $R$ corresponding to the choice of a fundamental chamber $C \subset E^\circ = E \setminus \bigcup_{\alpha \in R} \mathrm{Ker}(\alpha)$

Last time we proved:

(i) $W$ is generated by $\{s_i = s_{\alpha_i}\}_{i \in I}$ - simple reflections
(Prop. 3 of Lecture 24)

(ii) $R = \bigcup_{i \in I} W \cdot \alpha_i$ ($W$-orbits of simple roots)

(iii) $W \curvearrowright$ Set of chambers is transitive
(Thm. 5 of Lecture 24)

## 2. Length function.-

$$\ell : W \longrightarrow \mathbb{Z}_{\geq 0}$$

(depending on the choice of $C \leftrightarrow \Delta$)

is defined as: $\ell(w) = \text{MIN}\{r : \exists\, i_1, \dots, i_r \in I \text{ s.t. } w = s_{i_1} \cdots s_{i_r}\}$

Thus $\ell(w) = 0 \iff w = e$.

Ex. If $w \in W$ and $i \in I$, then $\ell(ws_i) = \ell(w) \pm 1$.

Theorem.- Let $w \in W$ and $i \in I$. Then, the following are equivalent.

(1) $\ell(ws_i) < \ell(w)$

(2) $w(\alpha_i) \in R_-$

(3) - Exchange property - For every reduced expression $w = s_{i_1} \cdots s_{i_\ell}$ (i.e., $\ell = \ell(w)$)

$\exists\, 1 \leq j \leq \ell$ s.t. $s_{i_j} \cdots s_{i_\ell} = s_{i_{j+1}} \cdots s_{i_\ell} \cdot s_i$

In particular $w = s_{i_1} \cdots \widehat{s_{i_j}} \cdots s_{i_\ell} s_i$ is another reduced expression of $w$. ($\widehat{s_{i_j}}$: skipped)

Proof.- (3) $\Rightarrow$ (1). Using the fact that $w$ has a (reduced) expression terminating in $s_i$, and $s_i^2 = e$,

$ws_i$ has smaller length than $w$.

(2) $\Rightarrow$ (3). Let $w = s_{i_1} \cdots s_{i_\ell}$ be a reduced expression and $i \in I$ so that $w(\alpha_i) \in R_-$. Consider the sequence of roots

$$\underset{\in R_+}{\alpha_i} \qquad s_{i_\ell}(\alpha_i) \qquad s_{i_{\ell-1}} s_{i_\ell}(\alpha_i) \qquad \cdots \qquad \underset{\in R_-}{s_{i_1} \cdots s_{i_\ell}(\alpha_i) = w(\alpha_i)}$$

Let $1 \leq j \leq \ell$ be such that

$$\beta = s_{i_{j+1}} \cdots s_{i_\ell}(\alpha_i) \in R_+ \qquad s_{i_j}(\beta) \in R_-.$$

[ Lemma: $\alpha \in R_+$ and $s_j(\alpha) \in R_- \Rightarrow \alpha = \alpha_j$

Proof of the lemma.- Write $\alpha = \sum_r n_r \alpha_r$ and hence

$$s_j(\alpha) = \alpha - \alpha(h_j)\alpha_j$$
$$= \sum_{r \neq j} n_r \alpha_r + (n_j - \alpha(h_j))\alpha_j$$

If $\alpha \neq \alpha_j$, $\exists\ r \neq j$ s.t. $n_r \geq 1$.

Hence, $s_j(\alpha) \in R_+$ by sign coherence. □ ]

This lemma implies that $\beta = \alpha_{i_j}$

$$s_{i_{j+1}} \cdots s_{i_\ell}(\alpha_i) = \alpha_{i_j} \qquad \left( \text{recall } \beta = u(\alpha) \Rightarrow s_\beta = u s_\alpha u^{-1} \right)$$

$$\Rightarrow s_{i_{j+1}} \cdots s_{i_\ell} s_i = s_{i_j} s_{i_{j+1}} \cdots s_{i_\ell}.$$

which is what we wanted to show.

Thus, we have proved (2) $\Rightarrow$ (3) $\Rightarrow$ (1), i.e.

$$w(\alpha_i) \in R_- \Rightarrow \ell(ws_i) < \ell(w)$$

(1) $\Rightarrow$ (2). If $\ell(ws_i) < \ell(w)$, let $u = ws_i$ then $\ell(u) < \ell(us_i)$. Contrapositive of the implication proved above implies that $u(\alpha_i) \in R_+$

Hence $w(\alpha_i) = u(s_i(\alpha_i)) = -u(\alpha_i) \in R_-$ □

3. Corollary. - [W-action on the set of chambers is transitive]

If $x, y \in \overline{C}$ ($x, y \in C$) and $w(x) = y$ then $x = y$. ($w = e$ and hence $x = y$).

Proof.- By induction on $\ell(w)$. If $\ell(w) = 0$, $w = e$ and we are done

Assume $\ell = \ell(w) \geq 1$ and let $w = s_{i_1} \cdots s_{i_\ell}$ be a reduced expression. As $\ell(ws_{i_\ell}) < \ell(w)$, by the theorem above, $w(\alpha_{i_\ell}) \in R_-$. Hence $w(\alpha_{i_\ell})(y) \leq 0$ (resp. $< 0$ if $y \in C$)

But $w(\alpha_{i_\ell})(y) = (w\alpha_{i_\ell})(wx) = \alpha_{i_\ell}(x) \geq 0$ (resp $> 0$ if $x \in C$)

$\Rightarrow \alpha_{i_\ell}(x) = 0$ and hence $s_{i_\ell}(x) = x$ (resp. contradiction, hence $w = e$)

and $y = s_{i_1} \cdots s_{i_{\ell-1}}(x)$

By induction hypothesis. $x = y$ as claimed. □

## 4. Rank 2 relations. ( $A = (a_{ij})$ Cartan matrix)

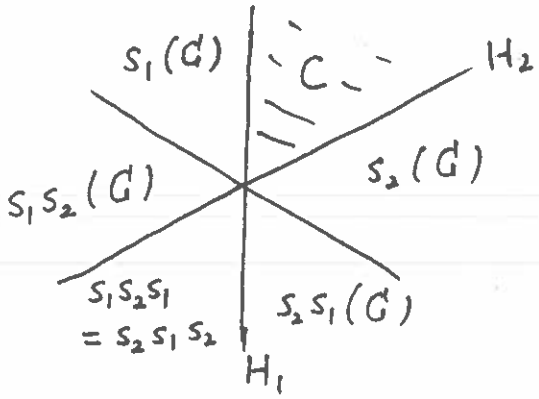
$A_1 \times A_1$ $\quad \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $\quad R = \{\pm\alpha_1, \pm\alpha_2\}$ $\quad \boxed{m = \text{ord}(s_1 s_2) = 2.}$

$s_1 s_2 = s_2 s_1$

$A_2$ $\quad \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ $\quad R_+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2\}$

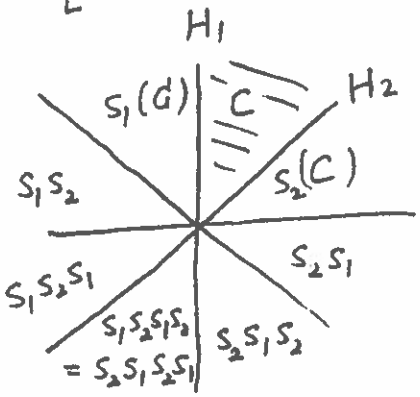
$s_1 s_2 s_1 = s_2 s_1 s_2$ $\quad \boxed{m = \text{ord}(s_1 s_2) = 3.}$

Hyperplane arrangement

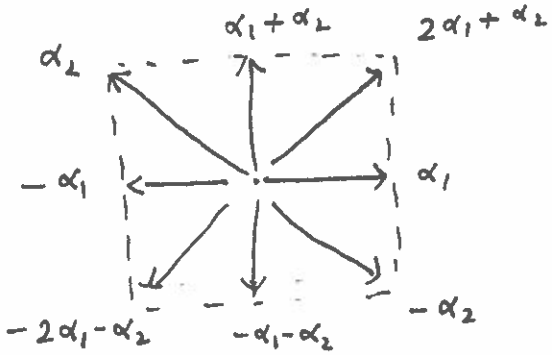



$B_2$ $\quad \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}$ $\quad R_+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2\}$

$s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1$ $\quad \boxed{m = \text{ord}(s_1 s_2) = 4.}$



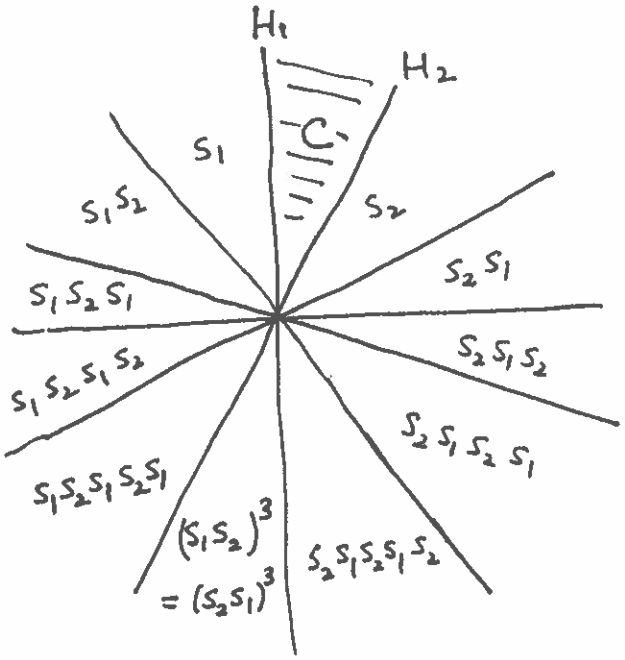

Hyperplane arrangement



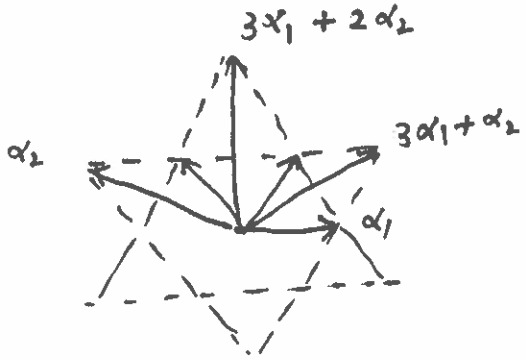

Roots

$G_2$ $\quad \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ $\quad R_+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2\}$

$s_1 s_2 s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1 s_2 s_1$ $\quad \boxed{m = 6}$




Hyperplane arrangement




Roots

5. Coxeter presentation - $\forall\ i \neq j$ in $I$, define

$$m_{ij} = 2, 3, 4 \text{ or } 6 \quad \text{if}$$

$$a_{ij}\, a_{ji} = 0, 1, 2 \text{ or } 3 \qquad (A = (a_{ij}) \text{ Cartan Matrix})$$

Theorem (Tits' lemma) - $W$ admits the following presentation: generators $\{s_i\}_{i \in I}$

Relations: $s_i^2 = e$

$$\underbrace{s_i s_j s_i \cdots}_{m_{ij} \text{ terms}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij} \text{ terms}} \qquad \forall\ i \neq j \text{ in } I$$

Proof - We have to prove the following - given a group $G$ and a set of elements $\{g_i\}_{i \in I} \subset G$ s.t. $g_i^2 = e\ \forall\ i$

and $g_i g_j g_i \cdots = g_j g_i g_j \cdots$ ($m_{ij}$ - terms)

There is a unique group hom. $\varphi : W \to G$ so that

$$\varphi(s_i) = g_i \quad \forall\, i \in I.$$

Define $\varphi$ as follows. - let $w = s_{i_1} \cdots s_{i_\ell}$ be a reduced expression

set $\varphi(w) := g_{i_1} \cdots g_{i_\ell}$

Ex. - Assuming $\varphi$ is well-defined, show that it is a group hom.

In order to prove that the product $g_{i_1} \cdots g_{i_\ell} \in G$ is independent of the choice of the reduced expression* $s_{i_1} \cdots s_{i_\ell}$ of $w$, we proceed by induction on $\ell = \ell(w)$.

The base cases $\ell = 0, 1$ are obvious. ($w = e$ or $w = s_i$ is the only red. exp.)

Now let $R(w)$ = set of reduced expressions of $w$.

$$= \{ \underline{i} = (i_1, \ldots, i_\ell) \in I^\ell : w = s_{i_1} \cdots s_{i_\ell} \}$$

Note: if $\underline{i} = (i_1, \ldots, i_\ell)$ and $\underline{j} = (j_1, \ldots, j_\ell)$ are in $R(w)$

s.t. $i_1 = j_1$ or $i_\ell = j_\ell$ then $g_{i_1} \cdots g_{i_\ell} = g_{j_1} \cdots g_{j_\ell}$ by induction

denote $g(\underline{i}) = g_{i_1} \cdots g_{i_\ell}$

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* this argument is due to J. Tits, and only uses $g_i g_j g_i \cdots$ rel$^{n}$
$g_i^2 = 1$ is not needed

Let $\underline{i} = (i_1, \cdots, i_\ell)$ and $\underline{j} = (j_1, \cdots, j_\ell)$ be two red. exp. of $w$.

Using the exchange property (Thm. 2 (3) above), applied to

$w = s_{i_1} \cdots s_{i_\ell}$

$\ell(w s_{j_\ell}) < \ell(w)$

, we get another element of $R(w)$

$(i_1, \ldots, \hat{i_k}, \ldots, i_\ell, j_\ell)$

if $k \neq 1$, we are again done by induction.

$$g(\underline{i}) = g(i_1, \ldots, \hat{i}_k, \ldots, i_\ell, j_\ell) = g(\underline{j})$$

share $1^{st}$ term — share last term.

⟶ either we are done by induction, or the exchange prop always drops $k = 1$ term. In this case, continuing to use the exchange property gives a sequence of elements in $R(w)$

$i_1 \cdots i_\ell \overset{\ell(ws_{j_\ell}) < \ell(w)}{\rightsquigarrow} i_2\, i_3 \cdots i_\ell\, j_\ell \overset{\ell(ws_{i_\ell}) < \ell(w)}{\rightsquigarrow} i_3\, i_4 \cdots i_\ell\, j_\ell\, i_\ell$

$j_1 \cdots j_\ell \overset{ws_{i_\ell} < w}{\rightsquigarrow} j_2\, j_3 \cdots j_\ell\, i_\ell \rightsquigarrow j_3\, j_4 \cdots j_\ell\, i_\ell\, j_\ell$

$\rightsquigarrow \cdots i_\ell\, j_\ell\, i_\ell = \underline{i}'$ Now $g(i') = g(j')$ by assumption

...

$\rightsquigarrow \cdots j_\ell\, i_\ell\, j_\ell = \underline{j}'$

$g(\underline{i}) = g(\underline{i}') = g(\underline{j}') = g(\underline{j})$ and we are done

share last term