Lecture 27

Uniqueness of Cartan subalgebras

1. Let $\mathfrak{g}$ be a finite-dim'l, semisimple Lie algebra over $\mathbb{C}$.

For each $x \in \mathfrak{g}$, let $N(x) = \dim\left(\bigcup_{n \geq 1} \operatorname{Ker}(ad(x))^n \right)$

$$= \dim \{ y \in \mathfrak{g} : ad(x)^n \cdot y = 0 \text{ for some } n \geq 1 \}$$

Definition - $\operatorname{Rank}(\mathfrak{g}) := \operatorname{MIN} \{ N(x) : x \in \mathfrak{g} \}$

$\mathfrak{g}^0$ = set of generic (or "strongly regular" - or, later, "regular semisimple") elements

$$:= \{ x \in \mathfrak{g} : N(x) = \operatorname{Rank}(\mathfrak{g}) \}$$

Alternately, let $P(x;t) := \det(t \cdot Id_{\mathfrak{g}} - ad(x))$. This is a polynomial of degree $\dim(\mathfrak{g})$, whose coefficients are polynomial functions $\mathfrak{g} \to \mathbb{C}$. Note: $ad(x) \cdot x = 0 \Rightarrow ad(x)$ is not invertible $\Rightarrow P(x;0) = 0$.

By definition, $N(x)$ is the largest r s.t. t^r divides $P(x;t)$

Write: $P(x;t) = t^{\operatorname{rank}(\mathfrak{g})} \left(t^{\dim(\mathfrak{g}) - \operatorname{rank}(\mathfrak{g})} + \cdots + a_0(x) \right)$

Then x is generic $\iff a_0(x) \neq 0$.

In particular, we obtain the following.

Proposition - $\mathfrak{g}^\circ \subset \mathfrak{g}$ is open, dense and connected.

Proof. - $\mathfrak{g}^\circ = \{x : a_0(x) \neq 0\}$, where $a_0 : \mathfrak{g} \to \mathbb{C}$ is a non-constant poly. fn.

Thus, $\mathfrak{g} \setminus \mathfrak{g}^\circ$ is given by vanishing of a poly. function. so, it is closed of (complex) codim = 1.

$\Rightarrow$ $\mathfrak{g}^\circ$ is open & dense.

To see that $\mathfrak{g}^\circ$ is connected, let $x, y \in \mathfrak{g}^\circ$ and take $p(t) = a_0(tx + (1-t)y)$. As $p(t) = 0$ has finitely many isolated solutions, we can let $t \in \mathbb{C}$ be a path joining 0 to 1, avoiding this finite set - along which $tx + (1-t)y \in \mathfrak{g}^\circ$. □

2. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, $R \subset \mathfrak{h}^* \setminus \{0\}$ corresponding set of roots and $\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \right)$ root space decomposition.

Theorem. - (1) $\operatorname{Rank}(\mathfrak{g}) = \dim(\mathfrak{h})$

(2) $\mathfrak{g}^\circ \cap \mathfrak{h} = \mathfrak{h}^{reg} := \{h \in \mathfrak{h} : \alpha(h) \neq 0 \ \forall \alpha \in R\}$

"regular elements of $\mathfrak{h}$".

Remark $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$, $\mathfrak{h} = n$-dim'l, abelian subalg consisting of diagonal matrices

$$R = \{\varepsilon_i - \varepsilon_j : 1 \leq i \neq j \leq n+1\} \; ; \quad \varepsilon_i : \mathfrak{h} \longrightarrow \mathbb{C}$$

$$\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{n+1} \end{bmatrix} \longmapsto \lambda_i$$

In this case, $\mathfrak{h}^{reg} \simeq \mathbb{C}_0^{n+1} \setminus \bigcup_{i \neq j} (z_i = z_j)$

$\uparrow$

$\{ \lambda_1, \dots, \lambda_{n+1} : \sum_{j=1}^{n+1} \lambda_j = 0 \} \subset \mathbb{C}^{n+1}$.

subspace

——— end of remark ———

Proof of the theorem. – Let $\mathfrak{g}_{\neq 0} := \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \subset \mathfrak{g}$ linear subspace.

Note: $\forall\, x \in \mathfrak{h}^{reg}$, $ad(x) : \mathfrak{g}_{\neq 0} \to \mathfrak{g}_{\neq 0}$ is an isomorphism

More generally, for $h \in \mathfrak{h}$

$$Z_{\mathfrak{g}}(h) = \mathrm{Ker}(ad(h)) = \mathfrak{h} \oplus \bigoplus_{\alpha : \alpha(h) = 0} \mathfrak{g}_\alpha .$$

Let $G = \mathrm{Aut}_{\mathrm{LieAlg}}(\mathfrak{g}) \subset GL(\mathfrak{g})$ closed. In particular, we will work with "1-parameter subgroups": for $z \in \mathfrak{g}$ nilpotent

$$\exp(ad(tz)) \in G, \quad \forall\, t \in \mathbb{C}^*$$

Under the map $G \times \mathfrak{h}^{reg} \xrightarrow{\psi} \mathfrak{g}$, $(\sigma, x) \longmapsto \sigma(x)$, for a fixed $x \in \mathfrak{h}^{reg}$, $y \in \mathfrak{g}_\alpha$ $(\alpha \in R \setminus \{0\})$

$$\frac{d}{dt} \exp(t\, ad(y)) \cdot x = [y, x] = -\alpha(x)\, y$$

(non-zero scalar multiple of y)

$\Rightarrow$ Tangent space at $x \in \mathfrak{g}$ in $G \cdot x \subset \mathfrak{g}$, contains $\mathfrak{g}_{\neq 0}$

$$\Rightarrow \quad d\psi : T_{(1,x)}(G \times \mathfrak{z}^{reg}) \longrightarrow T_x(\mathfrak{g})$$

is surjective, hence $\mathrm{Image}(\psi) \supset$ an open neighbourhood $\mathcal{U}$ of x.

Since $\mathfrak{g}^\circ \subset \mathfrak{g}$ is dense, $\mathcal{U} \cap \mathfrak{g}^\circ \neq \phi$, i.e. we can find $g \in G$ and $y \in \mathfrak{g}^\circ$ s.t. $g(x) = y$. But then $Z_{\mathfrak{g}}(x)$ ~~and~~ has the same dimension as $N(y) = \mathrm{rank}(\mathfrak{g})$, and we are done showing $\mathrm{rank}(\mathfrak{g}) = \dim(\mathfrak{z})$ since $Z_{\mathfrak{g}}(x) = \mathfrak{z}$.

(2) is now obvious, given the Note on the previous page. □

3. Cartan subalgebras from generic elements.

Theorem.- Let $x \in \mathfrak{g}^\circ$ be semisimple. Then $Z_{\mathfrak{g}}(x)$ is a Cartan subalgebra of $\mathfrak{g}$.

Proof.- Consider the eigenspace decomposition of $\mathfrak{g}$ with respect to $\mathrm{ad}(x)$:

$$\mathfrak{g} = Z_{\mathfrak{g}}(x) \oplus \bigoplus_{\lambda \in \mathbb{C}\setminus\{0\}} \mathfrak{g}_\lambda \quad ; \quad \text{set } \mathfrak{g}_{\neq 0} := \bigoplus_{\lambda \neq 0} \mathfrak{g}_\lambda$$

We know (see Lecture 22, §2) that $Z_{\mathfrak{g}}(x)$ is reductive. Let us prove that it is nilpotent, hence abelian.

Consider $y \in Z_{\mathfrak{g}}(x)$ and $\mathrm{ad}(x+ty) : \mathfrak{g}_{\neq 0} \to \mathfrak{g}_{\neq 0}$ $(t \in \mathbb{C})$

(note that $[x,y]=0 \Rightarrow ad(y)$ preserves $\mathfrak{g}_\lambda \ \forall \lambda \in \mathbb{C}$).

Since $ad(x+ty): \mathfrak{g}_{\neq 0} \to \mathfrak{g}_{\neq 0}$ is invertible at $t=0$, it is invertible $\forall t$, $|t|<r$ (for some $r \in \mathbb{R}_{>0}$).

$\Rightarrow N(x+ty) \leq \dim Z_{\mathfrak{g}}(x) = \operatorname{rank}(\mathfrak{g})$ as $x \in \mathfrak{g}^0$ $(|t|<r)$

by minimality of $\operatorname{rank}(\mathfrak{g})$ it must be equal – i.e. $ad(x+ty)$ acts nilpotently on $Z_{\mathfrak{g}}(x)$. As $ad(x+ty) = t\,ad(y)$ on $Z_{\mathfrak{g}}(x)$, we conclude that $ad(y) \circlearrowright Z_{\mathfrak{g}}(x)$ is nilpotent.

Thus, by Engel's theorem, $Z_{\mathfrak{g}}(x)$ is nilpotent – hence abelian.

It remains to prove that every $y \in Z_{\mathfrak{g}}(x)$ is semisimple.

Note: $y \in Z_{\mathfrak{g}}(x)$, $y = y_s + y_N \Rightarrow y_s, y_N \in Z_{\mathfrak{g}}(x)$

(Jordan-Chevalley decomposition)

Therefore, it suffices to prove that

$y \in Z_{\mathfrak{g}}(x) \Rightarrow y = 0$.

$ad(y) \circlearrowright \mathfrak{g}$ nilpotent

For every $z \in Z_{\mathfrak{g}}(x)$, $\mathrm{ad}(y) \cdot \mathrm{ad}(z)$ commute - since $Z_{\mathfrak{g}}(x)$ is abelian - hence $\mathrm{ad}(y) \circ \mathrm{ad}(z)$ is again nilpotent $\Rightarrow K_{\mathfrak{g}}(y, z) = 0 \quad \forall\, z \in Z_{\mathfrak{g}}(x)$. By non-degeneracy of $K_{\mathfrak{g}}$ restricted to $Z_{\mathfrak{g}}(x)$, this implies that $y = 0$. □

4. Corollary.- Every $x \in \mathfrak{g}^{\circ}$ is semisimple

Proof.- If $x = x_S + x_N$ is the Jordan-Chevalley dec. of $x \in \mathfrak{g}^{\circ}$, then $N(x) = N(x_S) = \mathrm{rank}(\mathfrak{g})$. By the theorem above, $Z(x_S)$ is a Cartan subalgebra, $x_N \in Z_{\mathfrak{g}}(x_S)$ must be 0 - i.e. $x = x_S$ is semisimple. □

5. Corollary. [Uniqueness of Cartan subalgebras]

(a) Every Cartan subalgebra of $\mathfrak{g}$ is of the form $Z_{\mathfrak{g}}(x)$ for some $x \in \mathfrak{g}^{\circ}$.

(b) Any two Cartan subalgebras are conjugate to each other - i.e. $\forall\, \mathfrak{h}_1, \mathfrak{h}_2 \subset \mathfrak{g}$ Cartan subalg-s, $\exists$ $g \in G$ s.t. $g(\mathfrak{h}_1) = \mathfrak{h}_2$.

Proof.- (a) follows from Thms 2 & 3 above ($\mathfrak{h} = Z_{\mathfrak{g}}(x)$ for every $x \in \mathfrak{h}^{\mathrm{reg}}$)

(b) Consider the equivalence relation on $\mathfrak{g}^\circ$: for $x, y \in \mathfrak{g}^\circ$,

$$x \sim y \quad \text{if} \quad \exists\, g \in G \text{ s.t. } g(Z(x)) = Z(y).$$

Ex. - Let $x \in \mathfrak{g}^\circ$, $\mathfrak{h} = Z_{\mathfrak{g}}(x)$ Cartan subalgebra and

$$\psi : G \times \mathfrak{h}^{reg} \longrightarrow \mathfrak{g}^\circ$$ the map considered in the proof of Thm 2 above.

Then, $\{y \in \mathfrak{g}^\circ : y \sim x\} = \text{Image}(\psi)$

(eq. class of x)

So, every equivalence class is open (we showed $\text{Im}(\psi)$ is open in Thm 2 above); $\mathfrak{g}^\circ$ = disjoint union of equivalence classes, and $\mathfrak{g}^\circ$ is connected, imply that there is only one equivalence class. The claim of (b) now follows from (a) and Thm. 3 above.

□