# LIE ALGEBRAS: PROBLEM SET 1

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(1) Determine whether the following subspaces are Lie subalgebras or not.
   (a) $S \subset \mathfrak{gl}_n(\mathbb{R})$ the subspace of symmetric matrices.
   (b) $A \subset \mathfrak{gl}_n(\mathbb{R})$ the subspace of skew-symmetric matrices.
   (c) Let $V$ be a vector space over a field $k$, together with a non–degenerate, bilinear form $\beta : V \times V \to k$.

$$S_\beta := \{X \in \mathfrak{gl}(V) : \beta(X(v), w) + \beta(v, X(w)) = 0, \ \forall \ v, w \in V\} \subset \mathfrak{gl}(V)$$

(2) Let $\mathfrak{g}$ be a Lie algebra over a field $k$. Recall that we have a Lie algebra homomorphism $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ given by $\mathrm{ad}(x) : y \mapsto [x, y]$. Show that the image of ad is an ideal in the Lie algebra $\mathrm{Der}(\mathfrak{g})$.

(3) Let $\mathfrak{g}$ be a Lie algebra over a field $k$. Determine whether there is an isomorphism of Lie algebras $\mathfrak{g} \ltimes_{\mathrm{ad}} \mathfrak{g}$ (semidirect product of $\mathfrak{g}$ with itself via $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$) and $\mathfrak{g} \oplus \mathfrak{g}$.

(4) Let $\mathfrak{g}$ be a Lie algebra over a field $k$. Let $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$ denote the subspace spanned by $\{[x, y] : x, y \in \mathfrak{g}\}$. Show that $[\mathfrak{g}, \mathfrak{g}]$ is an ideal in $\mathfrak{g}$ (called the commutator ideal). Prove that $[\mathfrak{g}, \mathfrak{g}]$ is the smallest ideal $I$ so that $\mathfrak{g}/I$ is abelian.

(5) Consider the Lie algebra of $2 \times 2$ upper triangular matrices

$$\mathfrak{b}_2(k) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in k \right\} \subset \mathfrak{gl}_2(k)$$

and let $k^2$ be its natural 2–dimensional representation. Show that $\mathrm{End}_{\mathfrak{b}_2(k)}(k^2) = k \cdot \mathrm{Id}_{2\times 2}$.

(6) Show that the center of $\mathfrak{gl}_n(k)$ is $k \cdot \mathrm{Id}_{n\times n}$.

(7) Assume that the characteristic of $k$ does not divide $n$. Let $\mathfrak{sl}_n(k) \subset \mathfrak{gl}_n(k)$ denote the Lie algebra of matrices of trace zero. Prove that the center of $\mathfrak{sl}_n(k)$ is $\{0\}$.

(8) Show that the natural representation $k^n$ of $\mathfrak{gl}_n(k)$ is irreducible.

(9) Let $H$ be a Hopf algebra over a field $k$. Consider the following subspace (called the space of primitive elements)

$$\mathrm{Prim}(H) := \{x \in H : \Delta(x) = x \otimes 1 + 1 \otimes x\}$$

Show that $\mathrm{Prim}(H) \subset \mathrm{Lie}(H)$ is a Lie subalgebra.

(10) Assume $k$ is of characteristic 0, and $\mathfrak{g}$ is a Lie algebra over $k$. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Show that $\mathrm{Prim}(U(\mathfrak{g})) = \mathfrak{g}$.

If $\mathrm{char}(k) = p$ is non–zero, then show that for every $x \in \mathfrak{g}$, $x^p \in \mathrm{Prim}(U(\mathfrak{g}))$.

(11) Let $\mathfrak{g}$ be a Lie algebra over a field $k$, and let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. Prove that $U(\mathfrak{g})$ has no zero–divisors.

(12) Let $(A, \mathcal{F}_\bullet(A))$ be a filtered algebra. Let $\{g_i\}_{i\in I}$ be a graded basis of $\mathrm{gr}_{\mathcal{F}}(A)$. Consider arbitrary lift $\widetilde{g}_i$ of $g_i$, for each $i \in I$. That is, if

$$g_i \in \mathrm{gr}_n(A) = \mathcal{F}_n(A)/\mathcal{F}_{n-1}(A)$$

then, choose $\widetilde{g}_i \in \mathcal{F}_n(A)$ so that $g_i = \widetilde{g}_i \bmod \mathcal{F}_{n-1}(A)$. Show that $\{\widetilde{g}_i\}_{i\in I}$ is a basis of $A$.

(13) Let $V$ be an $N$–dimensional vector space over $k$, and let $\mathcal{L}_V$ be the free Lie algebra over $V$. Recall that $\mathcal{L}_V = \bigoplus_{n=1}^{\infty} \mathcal{L}_V[n]$ is a graded (over $\mathbb{Z}_{\geq 1}$) Lie algebra. Let $d_n := \dim(\mathcal{L}_V[n])$. Show that we have the following identity:

$$\prod_{n=1}^{\infty}(1-t^n)^{d_n} = 1 - Nt$$

Use this to prove that

$$d_n = \frac{1}{n}\sum_{d|n} \mu(d) N^{n/d}$$

where, $\mu : \mathbb{Z}_{\geq 1} \to \{-1, 0, 1\}$ is the Möbius function defined as follows. $\mu(1) = 1$, $\mu(m) = 0$ if there is a prime $p$ so that $p^2$ divides $m$, and finally for distinct primes $p_1, \ldots, p_r$, $\mu(p_1 \cdots p_r) = (-1)^r$.

(14) Let $B$ be three dimensional vector space over $\mathbb{Q}$, with basis $\{b_1, b_2, b_3\}$. Let $\mu : B \wedge B \to B$ be given by:

$$\mu(b_1 \wedge b_2) = b_2,\ \mu(b_1 \wedge b_3) = b_2, \text{ and } \mu(b_2 \wedge b_3) = b_1.$$

(a) Show that $(B, \mu)$ is not a Lie algebra (Jacobi identity does not hold).

(b) Let $A$ be a unital, associative algebra over $\mathbb{Q}$, and $\alpha : B \to A$ be a linear map satisfying

$$\alpha(\mu(b \wedge b')) = \alpha(b)\alpha(b') - \alpha(b')\alpha(b)$$

for every $b, b' \in B$. Show that $\alpha(b_1) = 0$.

(c) Conclude that there is no action of $B$ on the polynomial ring $P = \mathbb{Q}[z_1, z_2, z_3]$ satisfying the following conditions:

- For each $\ell \geq 0$, $i_1 \leq \cdots \leq i_\ell$ from $\{1, 2, 3\}$, and $i \in \{1, 2, 3\}$,

$$b_i \cdot (z_{i_1} \cdots z_{i_\ell}) = \begin{cases} z_i z_{i_1} \cdots z_{i_\ell} & \text{if } i \leq i_1 \\ z_i z_{i_1} \cdots z_{i_\ell} \bmod P_\ell & \text{generally} \end{cases}$$

  where $P_\ell$ denotes the subspace of polynomials of degree $\leq \ell$.
- $b \cdot (b' \cdot \xi) - b' \cdot (b \cdot \xi) = \mu(b, b') \cdot \xi$, for all $b, b' \in B$ and $\xi \in P$.

(15) [1] Let $k = \mathbb{F}_2$ be the field with 2 elements, $\{x_1, x_2, x_3\}$ the standard basis of $k^3$, and $K = \bigwedge^\bullet(k^3)$. Thus, $K$ is 8–dimensional, commutative algebra over $k$, with a basis consisting of square–free monomials in $\{x_1, x_2, x_3\}$.

[1] N. Bourbaki, *Lie groups and Lie algebras*, Chapter 1, §2, Problem 9.

Let $\mathfrak{g}$ be a Lie algebra over $K$, which is a free $K$–module of rank 6 with a basis $\{e_1, e_2, e_3, e_{12}, e_{23}, e_{13}\}$, and the Lie bracket is given as follows:

$$[e_1, e_2] = e_{12},\ [e_1, e_3] = e_{13},\ [e_2, e_3] = e_{23}$$

(note: we are in char 2, so $[x, y] = [y, x]$), and all the other Lie brackets are zero.

Let $u = x_1e_1 + x_2e_2 + x_3e_3 \in \mathfrak{g}$ and let $\mathfrak{h} \subset \mathfrak{g}$ be the ideal generated by $u$. Thus, as a $K$–module, $\mathfrak{h}$ is generated by the following four elements: $u$ and

$$[u, e_1] = x_2e_{12} + x_3e_{13},\ [u, e_2] = x_1e_{12} + x_3e_{23},\ [u, e_3] = x_1e_{13} + x_2e_{23}.$$

Let $v = x_1x_2e_{12} + x_1x_3e_{13} + x_2x_3e_{23}$.

(a) Show that $v \notin \mathfrak{h}$. *Hint: consider the linear form $\varphi : \mathfrak{g} \to K$, which maps $e_1, e_2, e_3 \mapsto 0$ and $e_{12} \mapsto x_3$, $e_{23} \mapsto x_1$, and $e_{13} \mapsto x_2$. Show that $\varphi$ vanishes on $\mathfrak{h}$, but is non–zero on $v$.*

(b) Let $A$ be an arbitrary unital associative algebra over $K$ and $f : \mathfrak{g} \to \mathrm{Lie}(A)$ be a Lie algebra homomorphism. Show that $f(v) = f(u)^2$.

(c) Conclude that the natural map $\mathfrak{g}/\mathfrak{h} \to U(\mathfrak{g}/\mathfrak{h})$ is not injective.