LIE ALGEBRAS: PROBLEM SET 2

SACHIN GAUTAM

(1) Let k be a field and let $n \in \mathbb{Z}_{\geq 2}$. Recall the subalgebras of $\mathfrak{gl}_n(k)$

$$\mathfrak{b}_n(k) = \{x \in \mathfrak{gl}_n(k) : x_{ij} = 0,\ \forall\ i > j\},\ \mathfrak{n}_n(k) = \{x \in \mathfrak{gl}_n(k) : x_{ij} = 0,\ \forall\ i \geq j\}$$

Show that $\mathfrak{b}_n(k)$ is solvable and $\mathfrak{n}_n(k)$ is nilpotent.

(2) Let $\mathfrak{g}$ be a nilpotent Lie algebra and $\mathfrak{b}$ a non–zero ideal in $\mathfrak{g}$. Show that $\mathcal{Z}(\mathfrak{g}) \cap \mathfrak{b} \neq \{0\}$.

(3) Let $\mathfrak{g}$ be a solvable Lie algebra satisfying the following condition: for $x, y \in \mathfrak{g}$, if $[[x, y], y] = 0$ then $[x, y] = 0$. Prove that $\mathfrak{g}$ is abelian.

(4) Let $\mathfrak{g}$ be the following 3–dimensional Lie algbera over $\mathbb{C}$: in a basis $\{x, y, z\}$ of $\mathfrak{g}$, the Lie bracket is given by:

$$[x, y] = z, \text{ and } z \text{ is central.}$$

(a) Show that $\mathfrak{g}$ is nilpotent.
(b) Let V be a finite–dimensional, irreducible representation of $\mathfrak{g}$. Prove that z acts as 0 and that $\dim(V) = 1$.
(c) Let $V = \mathbb{C}[t]$ with $\mathfrak{g}$–action given as follows:

$$x \cdot p(t) = p'(t),\ y \cdot p(t) = tp(t), \text{ and } z \cdot p(t) = p(t)\ ,$$

for any polynomial $p(t) \in \mathbb{C}[t]$ (here $p'(t)$ is the derivative of $p(t)$). Prove that V is irreducible.

(5) Consider the following infinite–dimensional, solvable Lie algebra over $\mathbb{C}$, denoted by $\mathcal{H}$ (for Heisenberg). It has a basis $\{a_n : n \in \mathbb{Z}\} \cup \{c, d\}$, in which the Lie bracket is given as follows:

$$c \text{ is central},\ [d, a_n] = na_n,\ [a_n, a_m] = \delta_{n+m,0} nc\ .$$

For each $\alpha \in \mathbb{C}$, let $\mathcal{F}_\alpha$ (Fock space) be the following representation of $\mathcal{H}$. As a vector space, $\mathcal{F}_\alpha = \mathbb{C}[p_r : r \in \mathbb{Z}_{\geq 1}]$ is a polynomial ring in infinitely many variables, with the $\mathcal{H}$–action given by: c acts as $-\alpha$, a_n acts as multiplication by p_n $(n \geq 1)$, a_{-n} acts as $n\alpha \frac{\partial}{\partial p_n}$, $(n \geq 1)$, a_0 acts as 0, and

$$d \cdot (p_{\lambda_1} \cdots p_{\lambda_\ell}) = \left(\sum_{j=1}^{\ell} \lambda_j \right) p_{\lambda_1} \cdots p_{\lambda_\ell}$$

For $\alpha \neq 0$, show that $\mathcal{F}_\alpha$ is an irreducible representation of $\mathcal{H}$.

(6) Let K be a field of characteristic 2. Prove that $\mathfrak{sl}_2(K)$ is nilpotent (in fact, isomorphic to the Lie algebra of (4)).

(7) Let $\mathfrak{g}$ be 2–dimensional Lie algebra on a basis $\{x, y\}$, with the Lie bracket $[x, y] = y$.
   (a) Show that the Killing form on $\mathfrak{g}$ is non–zero.
   (b) Show that any invariant form on $\mathfrak{g}$ has to be degenerate.
   (c) Prove that $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ is surjective (i.e., every derivation is inner).

(8) Let $\mathfrak{g}$ be the 2–dimensional Lie algebra from Problem (7). Let k be of characteristic p and let V be a p–dimensional vector space on which $\mathfrak{g}$ acts, in a basis $\{v_0, \ldots, v_{p-1}\}$, by the following formulae:

$$x \cdot v_j = j v_j, \quad y \cdot v_j = v_{(j+1) \bmod p} .$$

Show that V is an irreducible, faithful representation of $\mathfrak{g}$.

(9) Let $\mathfrak{g}$ be a finite–dimensional Lie algebra. Assume k is of characteristic zero and algebraically closed. Show that if $\mathfrak{g}$ has a faithful, irreducible representation, then $\mathfrak{g}$ is reductive. (Compare with Problem (8)).

(10) Recall that for a semisimple Lie algebra $\mathfrak{g}$, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \ltimes \mathbb{C}^2$. Thus, $\mathfrak{sl}_2(C)$ is a Lie subalgebra of $\mathfrak{g}$, $\mathbb{C}^2$ is an abelian Lie subalgebra of $\mathfrak{g}$. As a vector space $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}^2$, and the Lie bracket of $x \in \mathfrak{sl}_2(\mathbb{C})$ with $v \in \mathbb{C}^2$ is given by the standard action of $\mathfrak{sl}_2(\mathbb{C})$ on $\mathbb{C}^2$: $[x, v] = -[v, x] = x \cdot v$. Show that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, but $\mathfrak{g}$ is not semisimple.

(11) Let $\mathfrak{h} = \mathbb{C} \cdot x$ be 1–dimensional Lie algebra and $\mathfrak{g}'$ be an arbitrary, finite–dimensional, abelian Lie algebra (over $\mathbb{C}$). Let $u \in \mathrm{End}(\mathfrak{g}')$ and define $\mathfrak{g} := \mathfrak{h} \ltimes \mathfrak{g}'$, where $[x, y] = u(y)$, for all $y \in \mathfrak{g}'$.
   (a) Prove that $\mathfrak{g}$ is nilpotent if, and only if u is a nilpotent operator.
   (b) Prove that the Killing form on $\mathfrak{g}$ is identically zero if, and only if, $\mathrm{Tr}(u^2) = 0$.
Give an example of a non–nilpotent Lie algebra whose Killing form is zero.

(12) Let E be a finite–dimensional vector space over k. Let $u, v \in \mathrm{End}(E)$ and let $\lambda \in k$ be an eigenvalue of u. Consider the generalized eigenspace

$$E^\lambda := \bigcup_{n=0}^{\infty} \mathrm{Ker}(u - \lambda \mathrm{Id})^n = \{x \in E : (u - \lambda)^N x = 0, \text{ for } N \gg 0\}$$

Assume that there is $n \in \mathbb{Z}_{\geq 1}$ such that $\mathrm{ad}(u)^n \cdot v = 0$. Show that v preserves E^λ.

(13) Assume that k is algebraically closed, but not necessarily of characteristic zero. Let V be a finite–dimensional k–vector space and $\mathfrak{g} \subset \mathfrak{gl}(V)$ a nilpotent, Lie subalgebra. Show that there is a decomposition of V, as a representation of $\mathfrak{g}$: $V = \bigoplus_{i=1}^{m} V(\lambda_i)$, where $\lambda_1, \ldots, \lambda_m : \mathfrak{g} \to k$ are functions (not necessarily linear forms), and for each $x \in \mathfrak{g}$, $1 \leq i \leq m$, the action of x on $V(\lambda_i)$ is of the form $\lambda_i(x) + u_i(x)$ with $u_i(x)$ being a nilpotent operator on $V(\lambda_i)$.

Take the example of $\mathfrak{g} = \mathfrak{sl}_2(k) \subset \mathfrak{gl}_2(k)$, where k is of characteristic 2, and $V = k^2$. Show that $m = 1$ and the corresponding $\lambda : \mathfrak{g} \to k$ is not a linear form.

(14) If k is algebraically closed and of characteristic zero, and V is a finite–dimensional representation of a nilpotent Lie algebra $\mathfrak{g}$, then show that $V = \bigoplus_{\lambda \in \mathfrak{g}^*} V^\lambda$ as $\mathfrak{g}$–representations, where, for each $\lambda \in \mathfrak{g}^*$, the generalized eigenspace V^λ is defined as

$$V^\lambda := \{v \in V : \ \forall\ x \in \mathfrak{g},\ (x - \lambda(x)\mathrm{Id})^N \cdot v = 0, \quad \text{for } N \gg 0\}$$

(15) Let $\mathfrak{g}$ be a finite–dimensional Lie algebra and let σ be a Lie algebra automorphism of $\mathfrak{g}$. Assume that k is algebraically closed and let $\mathrm{Spec}(\sigma) \subset k^\times$ be the set of eigenvalues of σ. For each $\lambda \in \mathrm{Spec}(\sigma)$, consider the generalized eigenspace

$$\mathfrak{g}^\lambda := \{x \in \mathfrak{g} : (\sigma - \lambda)^N(x) = 0,\ \text{for } N \gg 0\}\ .$$

Show that $\mathfrak{g} = \oplus_\lambda \mathfrak{g}^\lambda$ as vector space and $[\mathfrak{g}^\lambda, \mathfrak{g}^\mu] \subset \mathfrak{g}^{\lambda\mu}$. Assume that $\mathrm{Spec}(\sigma)$ does not contain any roots of unity, and show that $\mathfrak{g}$ is nilpotent.

(16) Again, let $\mathfrak{g}$ be a finite–dimensional Lie algebra and D be a derivation of $\mathfrak{g}$. Assume k is algebraically closed and let $\mathrm{Spec}(D) \subset k$ be the set of eigenvalues of D. Write the generalized eigenspace decomposition of $\mathfrak{g}$ as in the previous problem and show that $[\mathfrak{g}^\lambda, \mathfrak{g}^\mu] \subset \mathfrak{g}^{\lambda+\mu}$. Assuming k is of characteristic zero, and the adjoint action of $\mathfrak{h} := \mathfrak{g}^0$ on $\mathfrak{g}$ is nilpotent, show that $\mathfrak{g}$ is nilpotent.