LIE ALGEBRAS: PROBLEM SET 3

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(1) Let k be an arbitrary field and let V be a finite–dimensional k–vector space. Let $A \in \mathrm{End}(V)$ be a linear operator and $\mu(x) \in k[x]$ its minimal polynomial. Consider the decomposition $\mu(x) = f_1(x)^{n_1} \cdots f_\ell(x)^{n_\ell}$ into monic irreducible polynomials. Show that $V = \bigoplus_{i=1}^{\ell} V_i$, where $A(V_i) \subset V_i$, for each $1 \leq i \leq \ell$, and the minimal polynomial of $A|_{V_i}$ is $f_i(x)^{n_i}$. Moreover, if $\pi_i \in \mathrm{End}(V)$ is the canonical projection of V onto its i-th summand, then each π_i is a polynomial in A.

(2) Assume that k is a perfect field. Let $f(x) \in k[x]$ be a monic, irreducible polynomial, and let $n \in \mathbb{Z}_{\geq 1}$. Show that there exists $p(x) \in k[x]$ such that (a) $f(p(x)) = 0$ modulo $f(x)^n$; and (b) $(x - p(x))^n = 0$ modulo $f(x)^n$. Use this to prove, in the setting of the previous problem, that there is a polynomial $P(x)$ such that $P(A)$ has minimal polynomial $f_1(x) \cdots f_\ell(x)$ (hence, is semisimple) and $A - P(A)$ is nilpotent.

(3) Let k be an arbitrary field. Let V be a finite–dimensional k–vector space and $x \in \mathrm{End}(V)$ a nilpotent operator.

(a) Let $v \in V$ be a non–zero vector, and let $r \in \mathbb{Z}_{\geq 1}$ be smallest such that

$$x^r(v) = 0, \text{ and } x^{r-1}(v) \neq 0 \ .$$

Show that $\{v, x(v), \ldots, x^{r-1}(v)\}$ are linearly independent.

(b) For each $j \geq 0$, define $V_j := \mathrm{Ker}(x^j)$. Thus, we have a chain of subspaces (here n is smallest such that $x^n = 0$)

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

Show that $x(V_j) \subset V_{j-1}$, for all $1 \leq j \leq n$ and the induced map $x : V_j/V_{j-1} \to V_{j-1}/V_{j-2}$ is injective, for all $j \geq 2$.

(c) With the notations of the previous part, let $d_j := \dim(V_j/V_{j-1})$. Thus, $d_n \leq d_{n-1} \leq \cdots \leq d_1$ and $\sum_j d_j = N = \dim(V)$. Show that there is a basis of V in which x is block diagonal

$$J_1^{\oplus(d_1-d_2)} \oplus \cdots \oplus J_{n-1}^{\oplus(d_{n-1}-d_n)} \oplus J_n^{\oplus d_n}$$

Here, for each $\ell \geq 1$, J_ℓ denotes $\ell \times \ell$ Jordan matrix: $(J_\ell)_{i,i+1} = 1$, for all $1 \leq i \leq \ell - 1$, and the rest of the entries are zero.

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(4) Let $\mathfrak{g}$ be a simple, finite–dimensional Lie algebra over $\mathbb{C}$. Show that the space of invariant forms on $\mathfrak{g}$ is one dimensional. If $\mathfrak{a} = \mathfrak{g}_1 \oplus \cdots \mathfrak{g}_\ell$ is semisimple, with $\mathfrak{g}_j$ being its simple ideals, then show that the space of invariant forms on $\mathfrak{a}$ is ℓ–dimensional. Use this to show that the space of invariant, bilinear forms on $\mathfrak{a}$ is spanned by $\{\beta_V\}$, where V runs over irreducible, finite–dimensional, representations of $\mathfrak{a}$.

(5) Let $\mathfrak{g}$ be a finite–dimensional, simple Lie algebra over $\mathbb{C}$, $M = \mathfrak{g}^*$ with the natural $\mathfrak{g}$–action. Viewing M as an abelian Lie algebra, form the semidirect product $\widetilde{\mathfrak{g}} := \mathfrak{g} \ltimes M$. Let $\beta : \widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}} \to \mathbb{C}$ be defined as: for every $a, b \in \mathfrak{g}$ and $\xi, \eta \in M = \mathfrak{g}^*$

$$\beta(a + \xi, b + \eta) = \eta(a) + \xi(b)$$

Show that β is a non–degenerate, invariant form on $\widetilde{\mathfrak{g}}$. Show that there is no finite–dimensional representation V such that $\beta = \beta_V$.

(6) [1] In this problem, k is an algebraically closed of arbitrary characteristic.
   (a) Assume that $\mathfrak{g}$ is a nilpotent, finite–dimensional Lie algebra over k. Let V be a finite–dimensional representation of $\mathfrak{g}$, and $\beta_V : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ the associated invariant form. Show that $\beta_V(a, b) = 0$, for all $a \in \mathfrak{g}$ and $b \in \mathcal{Z}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$. Show that, if β_V is non–degenerate, then $\mathfrak{g}$ is abelian.
   (b) Now let $\mathfrak{g} = \mathfrak{gl}_2(k)$, with characteristic of k being 2. Take $V = k^2$, the standard vector representation of $\mathfrak{g}$. Show that $\mathfrak{g}$ is solvable and β_V is non–degenerate.

(7) Let $\mathfrak{g}$ be 6–dimensional $\mathbb{C}$–vector space, with a basis $\{a, b, c, d, e, f\}$. The following gives $\mathfrak{g}$ a structure of a Lie algebra

$$[a, b] = -[b, a] = d,\ [a, c] = -[c, a] = e,\ [b, c] = -[c, b] = f$$

and all the other Lie brackets among the basis vectors are 0. Consider the symmetric bilinear form $\beta : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ given by

$$\beta(c, d) = 1,\ \beta(a, f) = 1,\ \beta(b, e) = -1$$

and rest all are zero. Show that β is invariant and non–degenerate. Prove that $\mathfrak{g}$ is nilpotent, with $\mathcal{Z}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$. Conclude that β is not associated to any representation of $\mathfrak{g}$.

(8) Let $\mathfrak{g}$ be a finite–dimensional Lie algebra such that for every $x, y \in \mathfrak{g}$, $[[x, y], y] = 0 \Rightarrow [x, y] = 0$. Show that $\mathfrak{g}$ is reductive.

(9) Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$, $U = \mathcal{U}(\mathfrak{g})$ its enveloping algebra. We have a $\mathfrak{g}$–action on U, obtained from the adjoint action of $\mathfrak{g}$ on itself. Thus, for $x \in \mathfrak{g}$ and a monomial $y_1 \cdots y_r \in U$

$$x \cdot (y_1 \cdots y_r) = \sum_{i=1}^{r} y_1 \cdots y_{i-1}[x, y_i]y_{i+1} \cdots y_r$$

The linear dual $U^* = \mathrm{Hom}_{\mathbb{C}}(U, \mathbb{C})$ is naturally viewed as a $\mathfrak{g}$–representation. Show that

$$(U^*)^{\mathfrak{g}} = \{f \in U^* : f(ab) = f(ba),\ \forall\ a, b \in U\}$$

(10) Let V be a finite–dimensional vector space over $\mathbb{C}$ and $\mathfrak{g} = \mathfrak{sl}(V)$. For each $n \in \mathbb{Z}_{\geq 1}$, let $\pi_n : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}(V^{\otimes n})$ be the algebra homomorphism obtained from the natural action of $\mathfrak{g}$ on $V^{\otimes n}$. Show that for every $0 \neq z \in \mathcal{U}(\mathfrak{g})$, there exists n such that $\pi_n(z) \neq 0$.

[1] N. Bourbaki, *Lie groups and Lie algebras*, Chapter 1, §6, Ex. 22

Take $\mathfrak{sl}_2(k)$ with characteristic of k being 2. Show that $\pi_n(h^2 - h) = 0$ for all n. Here, $h = \mathrm{Id}_2 \in \mathfrak{sl}_2(k)$.

(11) Let $\mathfrak{g}$ be a finite–dimensional, semisimple Lie algebra over $\mathbb{C}$. Assume $x \in \mathfrak{g}$ is such that $\mathrm{ad}(x)$ is semisimple. For each $\alpha \in \mathbb{C}$, consider the eigenspace:

$$\mathfrak{g}[\alpha] := \{y \in \mathfrak{g} : [x, y] = \alpha y\}$$

Let $K_{\mathfrak{g}}$ be the Killing form of $\mathfrak{g}$. Show that $K_{\mathfrak{g}}(\mathfrak{g}[\alpha], \mathfrak{g}[\beta]) = 0$, if $\alpha + \beta \neq 0$. Prove that $\mathfrak{g}[0]$ is reductive.

(12) Show that $H^1(\mathfrak{g}; \mathfrak{g}) \cong \mathrm{Der}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})$, where $\mathrm{Der}(\mathfrak{g})$ is the Lie algebra of derivations of $\mathfrak{g}$, and $\mathrm{Inn}(\mathfrak{g}) \subset \mathrm{Der}(\mathfrak{g})$ is the ideal of inner derivations.

(13) Compute $H^1(\mathfrak{g}; M)$, when $\mathfrak{g}$ is two–dimensional Lie algebra, spanned by x, y, with the Lie bracket $[x, y] = y$; and $M = \mathbb{C}^2$ is the 2–dimensional representation:

$$x \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ y \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

(14) Let $\mathfrak{g}$ be a finite–dimensional Lie algebra over an arbitrary field k. Let $\mathrm{Rep}_{\mathrm{fd}}(\mathfrak{g})$ be the category of finite–dimensional representations of $\mathfrak{g}$. For $M \in \mathrm{Rep}_{\mathrm{fd}}(\mathfrak{g})$, $M^{\mathfrak{g}} := \{m \in M : x \cdot m = 0,\ \forall\ x \in \mathfrak{g}\}$.

(a) Given a short exact sequence in $\mathrm{Rep}_{\mathrm{fd}}(\mathfrak{g})$

$$0 \longrightarrow M_1 \xrightarrow{\ f\ } M \xrightarrow{\ g\ } M_2 \longrightarrow 0$$

and $v \in M_2^{\mathfrak{g}}$, consider the map $d_v : \mathfrak{g} \to M_1$ described as follows. Choose $m \in M$ such that $g(m) = v$. Since v is $\mathfrak{g}$–invariant, for every $x \in \mathfrak{g}$, we have $g(x \cdot m) = 0$. Thus, there is a unique $d_v(x) \in M_1$ such that $f(d_v(x)) = x \cdot m$. Show that $v \mapsto d_v$ is a well–defined linear map (called connecting homomorphism) $\partial^0 : M_2^{\mathfrak{g}} \to H^1(\mathfrak{g}; M_1)$.

(b) Verify the exactness of the following sequence at the vertices $M_2^{\mathfrak{g}}$ and $H^1(\mathfrak{g}; M_1)$.

$$0 \longrightarrow M_1^{\mathfrak{g}} \xrightarrow{\ f\ } M^{\mathfrak{g}} \xrightarrow{\ g\ } M_2^{\mathfrak{g}} \xrightarrow{\ \partial^0\ } H^1(\mathfrak{g}; M_1) \longrightarrow H^1(\mathfrak{g}; M) \longrightarrow H^1(\mathfrak{g}; M_2)$$

(15) Retain the notations of the previous problem. Let $0 \to V_1 \to V \to V_2 \to 0$ be a short exact sequence in $\mathrm{Rep}_{\mathrm{fd}}(\mathfrak{g})$, denoted by ξ below. Take

$$M_1 := \mathrm{Hom}_k(V_2, V_1),\ M := \mathrm{Hom}_k(V_2, V),\ \text{ and } M_2 := \mathrm{Hom}_k(V_2, V_2)$$

in the previous problem, and consider the connecting homomorphism constructed there

$$\partial^0 : \mathrm{Hom}_{\mathfrak{g}}(V_2, V_2) \to H^1(\mathfrak{g}; M_1)\ .$$

Show that ξ is split if, and only if $\partial^0(\mathrm{Id}_{V_2}) = 0$.