LIE ALGEBRAS: PROBLEM SET 4

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Recall our notations for the representation theory of $\mathfrak{sl}_2(\mathbb{C})$. In the basis $\{h, e, f\}$, the Lie bracket of $\mathfrak{sl}_2(\mathbb{C})$ is given by

$$[h, e] = 2e,\ [h, f] = -2f,\ \text{and}\ [e, f] = h\ .$$

For $\lambda \in \mathbb{C}$, M_λ denotes the Verma module of highest weight λ. It admits a basis $\{\Omega_j^{(\lambda)}\}_{j\in\mathbb{Z}_{\geq 0}}$ (we omit the superscript (λ), if it is clear from the context) in which $\mathfrak{sl}_2(\mathbb{C})$–action is as follows.

$$e \cdot \Omega_j = (\lambda - j + 1)\Omega_{j-1},\ f \cdot \Omega_j = (j+1)\Omega_{j+1},\ \text{and}\ h \cdot \Omega_j = (\lambda - 2j)\Omega_j\ .$$

For $n \in \mathbb{Z}_{\geq 0}$, L_n denotes the $n+1$–dimensional, irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$. To distinguish from the Verma modules, below, its basis will be denoted by $\{\mathsf{v}_j^{(n)}\}_{0\leq j\leq n}$.

In the following problems, you will give a direct proof of the complete reducibility theorem for $\mathfrak{sl}_2(\mathbb{C})$. You are allowed to use the following facts, but not the complete reducibility theorem iteself.

(A) If V is a finite–dimensional, $\mathfrak{sl}_2(\mathbb{C})$–representation, then e and f act nilpotently on V.
(B) The set of isomorphism classes of irreducible, finite–dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$ is $\{L_n : n \in \mathbb{Z}_{\geq 0}\}$.
(C) Let $V \in \mathrm{Rep}_{\mathrm{fd}}(\mathfrak{sl}_2(\mathbb{C}))$. Then, for every $\mu \in \mathbb{Z}_{\geq 0}$, we have natural isomorphism of vector spaces

$$\mathrm{Hom}_{\mathfrak{sl}_2(\mathbb{C})}(L_\mu, V) \cong \{v \in V : e \cdot v = 0,\ \text{and}\ h \cdot v = \mu v\}$$

The latter will be denoted by $V^e[\mu]$ below.

(1) We begin by proving h acts semisimply on V with $\mathbb{Z}$ eigenvalues.
   (a) Prove that the following identity holds in $\mathcal{U}(\mathfrak{sl}_2(\mathbb{C}))$. For every $m, n \in \mathbb{Z}_{\geq 0}$,

$$e^{(m)}f^{(n)} = \sum_{p=0}^{\mathrm{Min}(m,n)} \begin{pmatrix} h - m + n \\ p \end{pmatrix} f^{(n-p)}e^{(m-p)}\ ,$$

   where, the divided power notation $x^{(r)} = \frac{x^r}{r!}$ is used, and $\begin{pmatrix} x \\ r \end{pmatrix} = \frac{x(x-1)\cdots(x-r+1)}{r!} \in \mathbb{Q}[x]$, for every $r \geq 0$.
   (b) Prove the following identity in $\mathcal{U}(\mathfrak{sl}_2(\mathbb{C}))[\![z]\!]$

$$\sum_{n=0}^{\infty} e^{(n)}f^{(n)}z^n = (1+z)^h \sum_{n=0}^{\infty} f^{(n)}e^{(n)}z^n$$

where, $(1+x)^a := \exp(a\log(1+x)) = \sum_{r=0}^{\infty} \begin{pmatrix} a \\ r \end{pmatrix} x^r$ (by Newton's binomial theorem).

(c) Show that if V is a finite–dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$, then $(1+z)^h$ is a rational, $\mathrm{End}(V)$–valued function of $z \in \mathbb{C}$. Conclude from this, that h acts semisimply on V, with $\mathbb{Z}$–eigenvalues.

(2) Let V be a representation of $\mathfrak{sl}_2(\mathbb{C})$. Assume that there exists $0 \neq v \in V$, and $\mu \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$, such that $fe \cdot v = 0$ and $h \cdot v = \mu v$. Show that V is necessarily infinite–dimensional.

(3) Let $V \in \mathrm{Rep}_{\mathrm{fd}}(\mathfrak{sl}_2(\mathbb{C}))$. Prove that the following $\mathfrak{sl}_2(\mathbb{C})$–intertwiner (from (C) above) is injective:

$$\psi : \bigoplus_{n \in \mathbb{Z}_{\geq 0}} L_n \otimes V^e[n] \to V \ .$$

For each $v \in V$, let $n(v) \in \mathbb{Z}_{\geq 0}$ be the smallest such that $e^{n(v)} \cdot v = 0$. Prove, by induction on $n(v)$, that every weight vector $v \in V[\mu]$ ($\mu \in \mathbb{Z}$) lies in the image of ψ. Thus, ψ is an isomorphism, and V is completely reducible.

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(4) Prove the following *Clebsch–Gordon rule*: for $m, n \in \mathbb{Z}_{\geq 0}$, we have

$$L_m \otimes L_n \cong \bigoplus_{j=0}^{\mathrm{Min}(m,n)} L_{m+n-2j} \ \text{ as } \ \mathfrak{sl}_2(\mathbb{C})\text{–representations.}$$

(5) Compute the decomposition of $\underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n-\text{folds}}$ into a direct sum of irreducible $\mathfrak{sl}_2(\mathbb{C})$–representations.

(6) Let V be a finite–dimensional vector space over $\mathbb{C}$, and $A \in \mathrm{End}(V)$ a nilpotent operator. Show that there is an $\mathfrak{sl}_2(\mathbb{C})$–action on V, $\rho : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V)$, such that $\rho(e) = A$. (This result is known as Jacobson–Morozov lemma.)

For each $n \in \mathbb{Z}_{\geq 1}$, let $J_n \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ be the nilpotent Jordan matrix. Compute the Jordan block decomposition of $J_m \otimes \mathrm{Id}_n + \mathrm{Id}_m \otimes J_n$ in $\mathrm{Mat}_{mn\times mn}(\mathbb{C})$.

(7) Show that $C = \frac{1}{2}h^2 + ef + fe$ is in the center of $\mathcal{U}(\mathfrak{sl}_2(\mathbb{C}))$. Let V be an $\mathfrak{sl}_2(\mathbb{C})$–representation, $0 \neq v \in V$ and $\lambda \in \mathbb{C}$ are such that: $h \cdot v = \lambda v$ and $e \cdot v = 0$. Show that $C \cdot v = \frac{\lambda(\lambda+2)}{2} v$. Moreover, if v generates V as $\mathcal{U}(\mathfrak{sl}_2(\mathbb{C}))$–module (recall, such representations are called highest–weight representations, with highest weight λ), then $C = \frac{\lambda(\lambda+2)}{2}\mathrm{Id}_V$.

Now assume that V and W are two highest–weight representations of $\mathfrak{sl}_2(\mathbb{C})$, with highest weights $\lambda, \mu \in \mathbb{C}$ respectively. Show that if there is a non–zero $\mathfrak{sl}_2(\mathbb{C})$–intertwiner $V \to W$, then either $\lambda = \mu$; or $\mu \in \mathbb{Z}_{\geq 0}$ and $\lambda = -\mu - 2$.

(8) Prove that, for every $\lambda \in \mathbb{Z}_{\geq 0}$, there is a non–split short–exact sequence of $\mathfrak{sl}_2(\mathbb{C})$–intertwiners.

$$0 \to M_{-\lambda-2} \to M_\lambda \to L_\lambda \to 0$$

(9) Let V be a finite–dimensional $\mathbb{C}$–vector space, $\beta : V \times V \to \mathbb{C}$ a bilinear form, $X \in \mathrm{End}(V)$, nilpotent operator such that $\beta(X(v), w) + \beta(v, X(w)) = 0$, for every $v, w \in V$. Let $g(t) := \exp(tX)$, for $t \in \mathbb{C}$. Show that $\beta(g(t)(v), g(t)(w)) = \beta(v, w)$, for every $v, w \in V$.

(10) Prove that $\mathfrak{sl}_{n+1}(\mathbb{C})$ is simple, for all $n \geq 1$.

The problems below are inspired by the following beautiful paper of P. Etingof and A. Varchenko : *Dynamical Weyl group and applications*, Advances in Mathematics, 2002.

(11) Let $V \in \mathrm{Rep}_{\mathrm{fd}}(\mathfrak{sl}_2(\mathbb{C}))$ and let $v \in V[k]$ be a weight vector of weight $k \in \mathbb{Z}$. Prove that, for all but finitely many $\lambda \in \mathbb{C}$, there exists a unique $\mathfrak{sl}_2(\mathbb{C})$–intertwiner $\Phi : M_\lambda \to M_{\lambda-k} \otimes V$ such that $\Phi(\Omega_0^{(\lambda)}) = \Omega_0^{(\lambda-k)} \otimes v$ modulo the subspace spanned by $\{\Omega_j^{(\lambda-k)} \otimes v : j \geq 1, v \in V\}$.

Thus, generically in λ, there is a vector space isomorphism denoted by $\Phi \mapsto \langle \Phi \rangle \in V[k]$,

$$\mathrm{Hom}_{\mathfrak{sl}_2(\mathbb{C})}(M_\lambda, M_{\lambda-k} \otimes V) \to V[k]$$

$$\begin{aligned} \langle \Phi \rangle &= \text{ Coefficient of } \Omega_0^{(\lambda-k)} \text{ in } \Phi(\Omega_0^{(\lambda)}) \\ &= \left\langle \Omega_0^{(\lambda-k)} \mid \Phi \mid \Omega_0^{(\lambda)} \right\rangle \text{ (physicists' notation for the previous line)} \end{aligned}$$

For $v \in V[k]$, the corresponding intertwiner is denoted by Φ_λ^v.

(12) Let $V, W \in \mathrm{Rep}_{\mathrm{fd}}(\mathfrak{sl}_2(\mathbb{C}))$. The *fusion operator* $J_{V,W}(\lambda)$ is a linear operator on $V \otimes W$ defined as follows. Let $v \in V[k]$ and $w \in W[\ell]$ be two weight vectors. Then, (see notations above)

$$J_{V,W}(\lambda) \cdot (v \otimes w) = \left\langle (\Phi_{\lambda-\ell}^v \otimes \mathrm{Id}_W) \circ \Phi_\lambda^w \right\rangle$$

Prove the following properties of $J_{V,W}(\lambda)$:

(a) $J_{V,W}(\lambda)$ is zero–weight, and unipotent in the weight ordering. Meaning,

$$J_{V,W}(\lambda) : V[k] \otimes W[\ell] \to \bigoplus_{p \in \mathbb{Z}_{\geq 0}} V[k-2p] \otimes W[\ell+2p]$$

with $p = 0$ summand being $\mathrm{Id}_{V[k] \otimes W[\ell]}$.

(b) $J_{V,W}(\lambda)$ is a rational function of λ, taking value $\mathrm{Id}_{V \otimes W}$ at $\lambda = \infty$.

(13) Let $V \in \mathrm{Rep}_{\mathrm{fd}}(\mathfrak{sl}_2(\mathbb{C}))$. Define the dynamical reflection operator $A_V(\lambda)$ on V as follows. Let $v \in V[k]$ be a weight vector, $\Phi_\lambda^v : M_\lambda \to M_{\lambda-k} \otimes V$ the corresponding intertwiner, for $\lambda \in \mathbb{Z}$, $\lambda \gg 0$. In this case, we have $M_{-\lambda-2} \subset M_\lambda$ and $M_{-\lambda+k-2} \subset M_{\lambda-k}$.

(a) Show that Φ^v_λ maps $M_{-\lambda-2}$ to $M_{-\lambda+k-2} \otimes V$.

$$A_V(\lambda) \cdot v := \left\langle \Omega^{(-\lambda+k-2)} \mid \Phi^v_\lambda \mid \Omega^{(-\lambda-2)} \right\rangle$$

(b) Compute $A_V(\lambda)$ for $V = L_1$. *A bit harder:* show that with $V = L_n$ and $v = \mathsf{v}_r$ $(0 \leq r \leq n)$, we have the following answer:

$$A_{L_n}(\lambda) \cdot \mathsf{v}_r = (-1)^r \prod_{j=1}^{r} \frac{\lambda + j + 1}{\lambda - n + r + j} \mathsf{v}_{n-r}$$

(14) Let $U, V \in \mathrm{Rep}_{\mathrm{fd}}(\mathfrak{sl}_2(\mathbb{C}))$. Prove the following equation, where $A_U(\lambda - \mathsf{w}_2) \otimes \mathrm{Id}_V : u \otimes v \mapsto A_U(\lambda - r)(u) \otimes v$, if $v \in V[r]$ is of weight r.

$$A_{U\otimes V}(\lambda) \circ J_{U,V}(\lambda) = J_{U,V}(-\lambda - 2) \circ (\mathrm{Id}_U \otimes A_V(\lambda)) \circ (A_U(\lambda - \mathsf{w}_2) \otimes \mathrm{Id}_V)$$