LIE ALGEBRAS: PROBLEM SET 5

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(1) Let $\mathfrak{a}$ be a finite–dimensional, semisimple Lie algebra over $\mathbb{C}$, $\mathfrak{a} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\ell$ as Lie algebras, where $\mathfrak{g}_i$ are the simple ideals of $\mathfrak{a}$.

For each $1 \leq i \leq \ell$, let $\mathfrak{h}_i \subset \mathfrak{g}_i$ be a Cartan subalgebra, $R_i \subset \mathfrak{h}_i^* \setminus \{0\}$ corresponding root system, $B_i \subset R_i$ a base, and A_i the associated Cartan matrix. Show that $\mathfrak{h} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_\ell$ is a Cartan subalgebra of $\mathfrak{a}$, and the root system R associated to $(\mathfrak{a}, \mathfrak{h})$ is disjoint union of R_i. Moreover $B = \cup_i B_i$ is a base of R and the Cartan matrix of (R, B) is block diagonal $A_1 \oplus \cdots \oplus A_\ell$.

(2) Let $\mathfrak{g}$ be a finite–dimensional, semisimple Lie algebra over $\mathbb{C}$, $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra and $R \subset \mathfrak{h}^* \setminus \{0\}$ the root system of the pair $(\mathfrak{g}, \mathfrak{h})$. Let $\alpha, \beta \in R$ be two non–proportional roots, and $\mathfrak{sl}_2^{(\alpha)} \subset \mathfrak{g}$ be an $\mathfrak{sl}_2(\mathbb{C})$–triple associated to the root α.

Show that $V = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{k\alpha+\beta}$ is an irreducible representation of $\mathfrak{sl}_2^{(\alpha)}$. Use this to prove that there exist $p, q \in \mathbb{Z}_{\geq 0}$ such that

$$\{k : k\alpha + \beta \in R\} = \{-p, -p+1, \ldots, q\}, \quad \text{is a segment.}$$

(3) Let $n \in \mathbb{Z}_{\geq 2}$ and $\mathfrak{g}$ be the Lie subalgebra of $\mathfrak{gl}_{2n}(\mathbb{C})$, consisting of the following elements:

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} : \ A, B, C, D \in \mathrm{Mat}_{n\times n}(\mathbb{C}), \quad B, C \text{ are symmetric} \right\}$$

Let $\mathfrak{h} \subset \mathfrak{g}$ consist of diagonal matrices (in $\mathfrak{g}$). Compute the root space decomposition of $\mathfrak{g}$ relative to $\mathfrak{h}$; find a base of this root system, and compute the Cartan matrix.

In problems 4 to 8, E is a finite–dimensional, $\mathbb{R}$–vector space together with a positive–definite form $(\cdot, \cdot) : E \times E \to \mathbb{R}$, inducing the isomorphism $\nu : E \xrightarrow{\sim} E^*$. $R \subset E^* \setminus \{0\}$ is a (finite) root system, $E^\circ = E \setminus \bigcup_{\alpha \in R} Ker(\alpha)$ corresponding hyperplane arrangement. We fix $C \subset E^\circ$ (fundamental chamber), let $R_\pm \subset R$ be the polarization it implies, and let $B = \{\alpha_i\}_{i \in I} \subset R_+$ the base. Let $h_i := \frac{2\nu^{-1}(\alpha_i)}{(\alpha_i, \alpha_i)}$ denote the simple coroots, $A = (a_{ij} = \alpha_j(h_i))_{i,j \in I}$ the Cartan matrix and W the Weyl group of R. Recall that W is generated by $\{s_i\}_{i \in I}$. The order of $s_i s_j$ is denoted by m_{ij}. $\ell : W \to \mathbb{Z}_{\geq 0}$ denotes the length function defined by the choice of simple reflections $\{s_i\}$.

(4) Define $Q := \sum_{\alpha \in R} \mathbb{Z}\alpha \subset E^*$ and $P := \{\gamma : \gamma(h_i) \in \mathbb{Z}\} \subset E^*$ (called the root and weight lattices respectively). Show that $Q \subset P$ and $|P/Q| = \det(A)$. Compute this determinant in type A_n.

(5) Let $w \in W$. Show that $\ell(w) = |\{\alpha \in R_+ : w(\alpha) \in R_-\}| = |R_+ \cap w^{-1}(R_-)|$. Prove that there is a unique $w_0 \in W$ of maximum possible length equal to $|R_+|$.

(6) Let $w = s_{i_1} \cdots s_{i_r}$ be a reduced expression (so $r = \ell(w)$). Define a sequence of roots:

$$\beta_1 = \alpha_{i_1},\ \beta_2 = s_{i_1}(\alpha_{i_2}),\ \ldots,\ \beta_r = s_{i_1} \cdots s_{i_{r-1}}(\alpha_{i_r})\ .$$

Prove that $\{\beta_j : 1 \leq j \leq r\} \subset R_+$, are all distinct and equal to the set $\{\alpha \in R_+ : w^{-1}(\alpha) \in R_-\}$.

In the special case, when $w = w_0$ the longest element (see Problem 5), $\{\beta_1, \ldots, \beta_N\} = R_+$ is a total order on the set of positive roots ($N = |R_+|$), depending on a reduced expression of w_0. Compute this order for the two reduced expressions $w_0 = s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1$ in B_2–type.

(7) Choose a total order on I, say $I = \{1, 2, \ldots, n\}$, and let $c := s_1 s_2 \cdots s_n$ (Coxeter element). Show that, if $x \in E$ is such that $c(x) = x$, then $x = 0$. That is, 1 is not an eigenvalue of c.

(8) Let $\mathrm{Aut}(R)$ consist of permutations $\sigma : R \to R$ preserving the positive–definite form. Let $\Omega = \mathrm{Aut}(A)$ denote the group of permutations $\pi : I \to I$ such that $a_{\pi(i),\pi(j)} = a_{ij}$ (symmetries of the Dynkin diagram of A).

Show that $\mathrm{Aut}(R) \cong \Omega \ltimes W$. Prove that $-\mathrm{Id} \in W$ in type E_8.

In Problems 9 to 14[1]*, W denotes a group generated by $S \subset W \setminus \{1\}$, and $s^2 = 1$ for every $s \in S$. Again, $\ell : W \to \mathbb{Z}_{\geq 0}$ is the associated length function, and we have the notion of reduced expressions $w = s_1 \cdots s_r$. For $s, t \in S$, $s \neq t$, we let $m(s,t)$ denote the order of st. Thus, $m(s,t) \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$.*

Consider the following three axioms on (W, S). Recall that we showed (E) implies (U) in class.

(P) There exists a family of subset $\{P_s \subset W\}_{s \in S}$ such that (a) $1 \in P_s,\ \forall s \in S$; (b) $P_s \cap sP_s = \emptyset,\ \forall s \in S$; (c) for every $w \in W$ and $s, t \in S$, if $w \in P_s$ and $wt \notin P_s$ then $sw = wt$.

(E) For $w \in W$, $s \in S$ such that $\ell(sw) < \ell(w)$, and an arbitrary expression $w = s_1 \cdots s_r$, there exists $0 \leq j \leq r-1$ such that

$$ss_1 \cdots s_j = s_1 \cdots s_{j+1}$$

(U) For any group G, and elements $\{g_s \in G\}_{s \in S}$ such that (i) $g_s^2 = 1,\ \forall s \in S$; and (ii) $(g_s g_t)^{m_{s,t}} = 1$ for every distinct $s, t \in S$, there exists a unique group homomorphism $\psi : W \to G$ satisfying $\psi(s) = g_s,\ \forall s \in S$.

(9) Assume the axiom (P). Show that (E) holds, and P_s consists precisely of elements $u \in W$ such that $\ell(su) > \ell(u)$.

[1]Almost verbatim taken from *N. Bourbaki, Lie groups and Lie algebras, Chapter 4, §1*

(10) Assume the axiom (E) and set $P_s := \{u : \ell(su) > \ell(u)\}$. Prove the axiom (P) with this family of subsets.

(11) Assume the axiom (U). Define $T := \{usu^{-1} : u \in W, s \in S\}$ (the set of reflections conjugate to elements of S). Prove that we have W–action on the set $\mathbb{Z}/2\mathbb{Z} \times T$ satisfying:

$$s \cdot (\varepsilon, t) = (\varepsilon(-1)^{\delta_{s,t}}, sts)$$

for every $s \in S$, $t \in T$, and $\varepsilon \in \{\pm 1\} = \mathbb{Z}/2\mathbb{Z}$ written multicatively.

(12) For every sequence $\mathbf{s} = (s_1, \ldots, s_r)$ of elements of S, consider the following sequence of elements of T (compare with Problem 6 above):

$$t_j = (s_1 \cdots s_{j-1}) s_j (s_{j-1} \cdots s_1), \ 1 \leq j \leq r$$

Let us denote this by $\Phi(\mathbf{s}) = (t_1, \ldots, t_r)$. For $t \in T$, let $n(\mathbf{s}; t) := |\{j : t_j = t\}|$. Show that $(-1)^{n(\mathbf{s};t)}$ depends only on the product $w = s_1 \cdots s_r$, denoted by $\eta(w; t)$. (Use previous problem).

Prove that $w = s_1 \cdots s_r$ is a reduced expression if, and only if, t_i's are distinct. In this case, $\{t_1, \ldots, t_r\} = \{t \in T : \eta(w; t) = -1\}$.

(13) Use problems 11 and 12 to prove (U) implies (E).

(14) Prove that the following pairs satisfy (E) or (P) and hence are Coxeter groups.

(a) $W = S_{n+1}$, permutation group on $n+1$ letters, and $S = \{s_i = (i\ i+1)\}_{1 \leq i \leq n}$.

(b) $W = S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$, $S = \{s_0, s_1, \ldots, s_n\}$, where $s_0 = (1, e_1)$ and for $1 \leq i \leq n-1$, $s_i = ((i\ i+1), 0)$. Here, $e_1, \ldots, e_n$ is the standard coordinate basis of $(\mathbb{Z}/2\mathbb{Z})^n$, viewed as vector space over $\mathbb{Z}/2\mathbb{Z}$.