

ALGEBRA I: MIDTERM I

Vanshika Khandwala

$$1) \quad G = GL_2(\mathbb{F}_3)$$

+	0 1 2	0 1 2
0	0 1 2	0 0 0
1	1 2 0	0 1 2
2	2 0 1	0 2 1

$$\begin{matrix} G & \times & G \\ g \cdot h & \rightarrow & ghg^{-1} \end{matrix}$$

$$x = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\text{stab}_G(x) = \{g \mid g \cdot x \cdot g^{-1} = x\}$$

Let

$$g \in \text{stab}_G(x)$$

$$g \cdot x \cdot g^{-1} = x$$

$$\Rightarrow g \cdot x = x \cdot g$$

$$\text{If } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a & 2b \\ c & 2d \end{pmatrix} = \begin{pmatrix} a & b \\ 2c & 2d \end{pmatrix}$$

$$\Rightarrow b = 2b \quad \text{and} \quad c = 2c$$

$$\Rightarrow b = c = 0$$

$$\Rightarrow \text{stab}_G(x) = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \neq 0, a, d \in \mathbb{F}_3 \right\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right\}$$

$$\Rightarrow |\text{stab}_G(x)| = (2 \cdot 2) = 4$$

$$\simeq 2/2^2 \times 2/2^2$$

all elements of
order 2.

$$Gx = \{g \cdot x \mid g \in G\} = \{g \cdot x \cdot g^{-1} \mid g \in G\}.$$

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ s.t. $ad - bc = 1$

$$g \times g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \begin{pmatrix} ad - 2bc & ab \\ -cd & 2ad - bc \end{pmatrix}$$

$$= \begin{pmatrix} 1 - bc & ab \\ -cd & 1 + ad \end{pmatrix}$$

Three possible cases :

$$ad = 0 \quad bc = 2 \quad \begin{pmatrix} 2 & ab \\ -cd & 1 \end{pmatrix} \quad : \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$$

5 ways

$$ad = 1 \quad bc = 0 \quad \begin{pmatrix} 1 & ab \\ -cd & 2 \end{pmatrix} \quad : \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$$

5 ways

$$ad = 2 \quad bc = 1 \quad \begin{pmatrix} 0 & ab \\ -cd & 0 \end{pmatrix} \quad : \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$$

2 ways

$$\therefore |G_x| = 12.$$

(if $y = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det y = ad - bc$ and all elements in G_x have $\det 1$ or 2)

because $\forall h \in G$ s.t. $\det h = 2$, $\exists g \in G$ s.t. $2h = g$ & $\det g = 1$
(multiplicative inverse of 2 is 2) $\Rightarrow h = 2g$ in \mathbb{F}_3

$$h \times h^{-1} = 2g \times 2g^{-1} = g \times g^{-1}.$$

$$\text{From Lemma 2.4 } |G_x| = |G_x| |\text{Stab}_G x| = 12 \times 4 = 48$$

which can be verified $|G_x|$ is # of ways selecting two linearly independent vector of length 2 in \mathbb{F}_3 i.e. $(3^2 - 1) \cdot (3^2 - 3)$

$$= 8 \times 6 = 48.$$

2) $\varphi: G \rightarrow G'$ group homomorphism.

$$\Sigma': G' = G'_0 \triangleright G'_1 \dots \triangleright G'_l = \{e\}$$

Let Σ be the sequence with terms

$$G_j = \varphi^{-1}(G'_j) \quad \forall 0 \leq j \leq l$$

and $G_{l+1} = \{e\}$

Proof: Σ is a composition series

$$G_0 = \varphi^{-1}(G'_0) = \varphi^{-1}(G')$$
$$= G$$

prove

$$\Sigma: G = G_0 \triangleleft G_1 \dots \triangleright G_l \triangleright G_{l+1} = \{e\}.$$

$$\forall 0 \leq j \leq l$$

let $g \in G_j$ & $h \in G_{j+1}$

$$\Rightarrow \exists g' \in G'_j \text{ & } h' \in G'_{j+1} \text{ s.t. } \varphi(g) = g' \quad \varphi(h) = h'$$

$$G'_{j+1} \triangleleft G'_j \Rightarrow g' h' (g')^{-1} \in G'_{j+1}$$

$$\Rightarrow \varphi(g) \cdot \varphi(h) \cdot \varphi(g)^{-1} \in G'_{j+1}$$

$$\Rightarrow \varphi(g h g^{-1}) \in G'_{j+1} \quad (\because \varphi \text{ is a homo.})$$

$$\Rightarrow g h g^{-1} \in G_{j+1}$$

Hence, $G_{j+1} \triangleleft G_j \Rightarrow \Sigma$ is a composition series

$\forall j \text{ in } 0, 1, \dots, l-1$

Define $\psi_j : g_{r_j}^{\varepsilon}(a) \rightarrow g_{r_j}^{\varepsilon'}(a')$

s.t. $x a_{j+1} \rightarrow \varphi(x) a'_{j+1}$

Claim: $\forall j$ ψ_j is an injective homomorphism.

Suppose $\exists x_1, x_2 \in G_j$ s.t.

$$x_1 a_{j+1} = x_2 a_{j+1} \Rightarrow x_2^{-1} x_1 \in a_{j+1}$$

$$\Rightarrow \varphi(x_2^{-1} x_1) \in a'_{j+1}$$

$$\psi_j(x_2 a_{j+1}) = \varphi(x_2) a'_{j+1}$$

$$= \varphi(x_2) \varphi(x_2^{-1} x_1) a'_{j+1}$$

$$= \varphi(x_2 x_2^{-1} x_1) a'_{j+1} \quad \varphi \text{ is a homo.}$$

$$= \varphi(x_1) a'_{j+1}$$

$$= \psi_j(x_1 a_{j+1}) \quad \text{Therefore, } \psi \text{ is well-defined}$$

Suppose $\exists x_1, x_2 \in G_j$ s.t. $x_1 a_{j+1} \neq x_2 a_{j+1}$

but $\psi_j(x_1 a_{j+1}) = \psi_j(x_2 a_{j+1}) \Rightarrow x_1^{-1} x_2 \notin a_{j+1}$

$$\Rightarrow \varphi(x_1) a'_{j+1} = \varphi(x_2) a'_{j+1}$$

$$\Rightarrow \varphi(x_1) \varphi(x_2) \in a'_{j+1}$$

$$\Rightarrow \varphi(x_1^{-1} x_2) \in a'_{j+1} \quad (\varphi \text{ is a homomorphism})$$

$$\Rightarrow x_1^{-1} x_2 \in a_{j+1} \quad \text{contradiction.}$$

$\therefore \psi$ is injective. \blacksquare

$\forall x_1, x_2 \in G_j$

$$\begin{aligned}\psi_j(x_1 c_{j+1} \dots x_2 c_{j+1}) &= \psi_j(x_1 x_2 c'_{j+1}) \\&= \varphi(x_1) \cdot \varphi(x_2) c'_{j+1} \\&= \varphi(x_1) c'_{j+1} \dots \varphi(x_2) c'_{j+1} \\&= \psi_j(x_1 c_{j+1}) \dots \psi_j(x_2 c_{j+1})\end{aligned}$$

$\Rightarrow \psi$ is a homomorphism.

$\therefore \forall 0 \leq j \leq l-1 \quad \psi_j$ is an injective group homomorphism. \checkmark

3.) G is a finite group. $K \triangleleft G$.

$P < G$ be a sylow - p - subgroup of G .

Prove: $G = K \cdot N_G(P)$

Claim 1 : Let $Q = K \cap P$, then Q is a Sylow - p - subgroup of K .

Proof of claim 1 :

$Q \subseteq K, P \Rightarrow Q$ is a p - subgroup of K .

Using Sylow's II theorem, $\exists Q' < K$, sylow - p - subgroup of K s.t. $Q \subseteq Q' \Rightarrow |Q| \leq |Q'|$

Since Q' is a p - subgroup of G , using Sylow's II theorem, $\exists g \in G$ s.t. $Q' \subseteq gPg^{-1}$.

Also, $Q' \subseteq K$

$\Rightarrow Q' \subseteq (gPg^{-1}) \cap K = gPg^{-1} \cap gKg^{-1}$ since K is normal

$\Rightarrow Q' \subseteq g(P \cap K)g^{-1}$

$\Rightarrow |Q'| \leq |g(P \cap K)g^{-1}| = |P \cap K| = |Q|$

$\Rightarrow |Q| = |Q'|$.

Q was a p - subgroup & Q' was a Sylow - p - subgroup of K

$\Rightarrow Q'$ is a Sylow - p - subgroup of K

Claim 2 : $G = K \cdot N_G(Q)$

Proof of claim 2 :

Let $g \in G$.

$$Q \subset K$$

$$g Q g^{-1} \subset g K g^{-1} = K \quad \text{because } K \text{ is normal}$$

Using Sylow's theorem (II), $g Q g^{-1}$ is a Sylow-p-subgroup of K , so $\exists k \in K$ s.t.

$$g Q g^{-1} = k Q k^{-1}$$

$$\Rightarrow k^{-1}g Q g^{-1}k = Q$$

$$\Rightarrow k^{-1}g \in N_G(Q)$$

$$\Rightarrow g \in K N_G(Q)$$

Hence, $G \subseteq K N_G(Q)$

$$\Rightarrow G = K N_G(Q) \quad \text{as } K N_G(Q) \subseteq G.$$

~~Exercise 11.5 (Ans 1)~~

$$G = K N_G(Q) \checkmark$$

Proof : $G \neq K N_G(P)$ in general

From theorem 11.5, we know A_5 is simple & it has more than one Sylow-5-subgroup P s.t. $N_G(P) \not\subseteq G$. Let $K = \{e\}$ is trivial normal subgroup, then $K N_G(P) = N_G(P) \not\subseteq G$.
 \therefore The given question is not true in general. !

4.) H and N are groups.

$\alpha, \beta: H \rightarrow \text{Aut}(N)$ group homomorphisms

$\exists T \in \text{Aut}(N)$ s.t. $\forall h \in H$

$$\alpha(h) = T \circ \beta(h) \circ T^{-1}$$

Prove: $H \times_{\alpha} N \simeq H \times_{\beta} N$

Define: $\varphi: H \times_{\alpha} N \rightarrow H \times_{\beta} N$

$$\text{s.t. } (h, x) \rightarrow (h, T^{-1}x) \checkmark$$

Claim: φ is an isomorphism.

φ is well-defined because T^{-1} is an automorphism of N .

Let $h_1, h_2 \in H$ & $x_1, x_2 \in N$

$$\varphi((h_1, x_1), (h_2, x_2)) = \varphi((h_1 h_2, \alpha(h_2^{-1})(x_1) \cdot x_2))$$

$$= \varphi((h_1 h_2, T \circ \beta(h_2^{-1}) \circ T^{-1}(x_1) \cdot x_2))$$

$$= (h_1 h_2, \beta(h_2^{-1})(T^{-1}(x_1)) \cdot T^{-1}(x_2))$$

T is a homomorphism &
 $T^{-1} \circ T = \text{Id}$.

$$\varphi((h_1, x_1)) \cdot \varphi((h_2, x_2)) = (h_1, T^{-1}x_1) \circ_{\beta} (h_2, T^{-1}x_2)$$

$$= (h_1 h_2, \beta(h_2^{-1})(T^{-1}x_1) \cdot (T^{-1}x_2))$$

\Rightarrow

$$\varphi((h_1, x_1) \cdot (h_2, x_2)) = \varphi((h_1, x_1)) \cdot \varphi((h_2, x_2))$$

$\therefore \varphi$ is a group homomorphism.

$$(h, x) \in \ker \varphi$$

$$\Rightarrow \varphi(h, x) = (e, e)$$

$$\Rightarrow (h, T^{-1}x) = (e, e) \Rightarrow h = e \\ T^{-1}x = e \Rightarrow x = e.$$

Hence, φ is injective.

$$\forall (h, x) \in H \times_{\beta} N \quad \exists (h, Tx) \in H \times_{\alpha} N \text{ and}$$

$$\varphi((h, Tx)) = (h, T^{-1}(Tx)) = (h, x).$$

Hence, φ is surjective.

$\therefore \varphi$ is an isomorphism

$$H \times_{\alpha} N \cong H \times_{\beta} N$$



5) G is a group s.t. $|G| = 162 = 2 \cdot 3^4$

Using Sylow's theorem (II).

$$n_3 \mid 2 \quad \text{and} \quad n_3 \equiv 1 \pmod{3} \Rightarrow n_3 = 1$$

$\Rightarrow \exists H \triangleleft G$ s.t. H is Sylow 3-subgroup of G

$$|H| = 81 \Rightarrow [G:H] = 2$$

Using (10.2)(II) Corollary (Every p-group is solvable)

H is solvable.

$|G/H| = 2 \Rightarrow G/H$ is abelian; hence solvable.

Using 10.3 proposition

$H \triangleleft G$ & $H, G/H$ are solvable

$\Rightarrow G$ is solvable.

BONUS : G is not nilpotent

\Rightarrow ~~Not~~ There exist a Sylow subgroup which is not normal.

(H.W 4, # 6
assigned problem
 \therefore not writing proof)

Construct G s.t. $\forall s \in G \setminus H, s^2 = e$ i.e.

take ~~s~~ s , element of order 2. Then

$$\forall h \in H \quad (sh)^2 = e \Rightarrow n_2 = 81 \Rightarrow G \text{ is not nilpotent.}$$

The above group is well defined as

$$G = H \sqcup sH$$

(H can be any group of order 81)

and $\forall h_1, h_2 \in H$ $[(sh_1)^2 = e \Rightarrow sh_1 sh_1 = e \Rightarrow sh_1 = h_1^{-1} s^{-1} = h_1^{-1}]$

$$sh_1 \cdot sh_2 = h_1^{-1} s \cdot sh_2 = h_1^{-1} h_2 \in H$$

\therefore Product of any two elements of sH is in H .