ALGEBRA 1: MID TERM 1

Instructions

- For full credit, you must present clear justifications of your claims.
- The use of Lecture Notes is permitted. You may use a result proved in the class, without including its proof. However, in that case, please cite the result clearly (e.g, write Theorem 8.2 if refering to Schrier's Theorem). You can consult the notes (for citations, or definitions) at https://people.math.osu.edu/gautam.42/F17/notes.html
- You should not ask friends, family members, classmates, Dr. Tsumura, myself or google for help.

Bon chance!

Problem 1. Let $G = \mathrm{GL}_2(\mathbb{F}_3)$. Consider the action of G on itself by conjugation: $g \cdot h = ghg^{-1}$. Consider the following element of G:

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Describe the orbit of X and its stabilizer subgroup.

Problem 2. Let $\varphi : G \to G'$ be a group homomorphism. Let Σ' be a composition series of G':

$$\Sigma' : G' = G'_0 \triangleright G'_1 \triangleright \cdots \triangleright G'_l = \{e\}$$

Let Σ be the sequence with terms $G_j = \varphi^{-1}(G'_j)$ for every $0 \leq j \leq l$; and $G_{l+1} = \{e\}$. Prove that Σ is a composition series of G. Prove that there are injective group homomorphisms $\mathrm{gr}_j^{\Sigma}(G) \to \mathrm{gr}_j^{\Sigma'}(G')$ for every $0 \leq j \leq l-1$.

Problem 3. Let G be a finite group, $K \triangleleft G$ be a normal subgroup and let $P < K$ be a Sylow subgroup of K. Prove that $G = K.N_G(P)$.

Problem 4. Let H and N be two groups and assume that there are two group homomorphisms $\alpha, \beta : H \to \mathrm{Aut}(N)$. Let us assume that there exists $T \in \mathrm{Aut}(N)$ such that for every $h \in H$ we have $\alpha(h) = T \circ \beta(h) \circ T^{-1}$. Prove that $H \ltimes_\alpha N$ is isomorphic to $H \ltimes_\beta N$.

Problem 5. Prove that every group of order 162 is solvable.
(**Bonus:** Give an example of a group of order 162 which is not nilpotent).