ALGEBRA 1: SAMPLE FINAL

Instructions

- For full credit, you must present clear justifications of your claims.
- If you are using a result, you must include a precise statement.

Problem 1. Let G be a finite group of order 224. Prove that G cannot be simple.

Problem 2. Let G be a finite group and V be a finite–dimensional representation (over $\mathbb{C}$) of G. Prove the following, where χ_V is the character of V.

$$\frac{1}{|G|}\sum_{g\in G}\chi_V(g) = \dim(V^G)$$

Problem 3. Let A be a commutative ring and M be an *Artinian* A–module. That is, for every descending chain of submodules $M_0 \supset M_1 \supset \cdots$ there exists $\ell \geq 0$ such that $M_\ell = M_{\ell+1} = \cdots$. Assume there is an *injective* A–linear map $f : M \to M$. Prove that f is an isomorphism.

Problem 4. Let A be a commutative ring and $\mathfrak{a} \subset A$ be an ideal satisfying the following property: $\mathfrak{a}$ is not finitely–generated, but every ideal properly containing $\mathfrak{a}$ is finitely–generated. Prove that $\mathfrak{a}$ is prime. Use this to show that A is Noetherian if, and only if every *prime* ideal is finitely–generated.

Problem 5. Let A be a commutative ring and $S \subset A$ be a multiplicatively closed set (we are assuming $1 \in S$ and $0 \notin S$). Let $j : A \to S^{-1}A$ be the natural ring homomorphism. Prove that if $\mathfrak{q} \subsetneq A$ is a primary ideal such that $\mathfrak{q} \cap S = \emptyset$, then $j^{-1}(S^{-1}\mathfrak{q}) = \mathfrak{q}$. Give an example, when $\mathfrak{q}$ is not primary, $\mathfrak{q} \cap S = \emptyset$ and $\mathfrak{q} \subsetneq j^{-1}(S^{-1}\mathfrak{q})$.