Homework 11

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(1) **Proof** Firstly, we have a natural bilinear mapping i from $(A/\mathfrak{a}, M)$ to $M/\mathfrak{a}M$, that is, sending $(\tilde{a}, m)$ to $\widetilde{am}$, where $\tilde{a}, \widetilde{am}$ is the image of a in $A/\mathfrak{a}$ and $am \in M$ repsectively, $a \in A, m \in M$. Suppose that $(\tilde{a}, m) = (\tilde{b}, n)$, then we have $a-b \in \mathfrak{a}$ and $m = n$. Then we know that $\widetilde{am} - \widetilde{bn} = (\widetilde{a-b})m = 0$ and hence this mapping is well defined. Besides, we can check that it is bilinear.

Let N be an A-module and we consider a bilinear mapping from $(A/\mathfrak{a}, M) \to N$. Then we prove that f can induce a mapping $\tilde{f} : M/\mathfrak{a}M \to N$ such that $\tilde{f} \circ i = f$. In fact, we only need to set $\tilde{f}(x + \mathfrak{a}M) = f(\tilde{1}, x)$, where $\tilde{1}$ is the image of 1 in $A/\mathfrak{a}$. We check that $\tilde{f}$ is well defined: let $x_1, x_2 \in M$ such that $x_1 - x_2 \in \mathfrak{a}M$ and hence $x_1 - x_2 = am$ for some $a \in \mathfrak{a}, m \in M$. Then we have:

$$f(\tilde{1}, x_1) - f(\tilde{1}, x_2) = f(\tilde{1}, x_1 - x_2) = f(\tilde{1}, am) = f(\tilde{a}, m) = f(0, m) = 0.$$

And we can check that:

$$\tilde{f} \circ i(\tilde{a}, m) = \tilde{f}(\widetilde{am}) = f(\tilde{1}, am) = f(\tilde{a}, m).$$

Therefore, $M/\mathfrak{a}M$ satisfies the university property of $A/\mathfrak{a} \otimes_A M$ and hence $M/\mathfrak{a}M \simeq A/\mathfrak{a} \otimes_A M$. □

(2) **Proof** For $m \in M$, we can write it as $p(m) + m - p(m)$. We can check that $p(m - p(m)) = p(m) - p^2(m) = p(m) - p(m) = 0$ and hence $m - p(m) \in M_1$ and $p(m) \in M_2$. Therefore, $M = M_1 + M_2$. Then we pick $m \in M_1 \cap M_2$. Since $m \in M_2$, we can find $n \in M$ such that $m = p(n)$. Notice that $m \in M_1$, we have:

$$0 = p(m) = p(p(n)) = p^2(n) = p(n) = m.$$

Therefore, this is a direct sum and hence $M = M_1 \oplus M_2$. □

(3) **Proof** We set $A = \mathbb{Z}$, $\mathfrak{a} = (m)$. Then by exercise (3), we know that $\mathbb{Z}_m \otimes_{\mathbb{Z}} N \simeq \mathbb{Z}_n/(m)\mathbb{Z}_n$. Then we only need to prove that $\mathfrak{Z}_n/(m)\mathbb{Z}_n \simeq \mathbb{Z}_d$, where $d = \gcd(m,n)$.

We consider a the natural mapping ϕ from $\mathbb{Z}_n$ to $\mathbb{Z}_d$, by sending the image of x (an integer) in $\mathbb{Z}_n$ to its image in $\mathbb{Z}_d$. Since $d|n$, we can find that the mapping is well defined and surjective. Then we prove $\operatorname{Ker}\phi = (m)\mathbb{Z}_n$. For $\widetilde{a} \in (m)\mathbb{Z}_n$, $m|a$ and hence the image of a in $\mathbb{Z}_d$ is 0. Then we consider $\widetilde{a} \in \mathbb{Z}$ such that $\phi(\widetilde{a}) = 0$. Then we know that $d|a$. Since $d = \gcd(m,n)$, we can find s,t such that $sm + nt = d$ and hence we have s', t' such that $s'm + t'n = a$. Then we can find $\widetilde{a} = \widetilde{s'm} = m\widetilde{s'}$ and hence $\operatorname{Ker}\phi \subseteq (m)\mathbb{Z}_n$. Therefore, we know that $\mathbb{Z}_n/(m)\mathbb{Z}_n \simeq \mathbb{Z}_d$, and hence $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n \simeq \mathbb{Z}_{\gcd(m,n)}$. □

(4) **Proof** We define the $f : \mathbb{Z}_m \to P_\alpha$ by sending $1_m \in \mathbb{Z}_m$ to $e_1 \in P_\alpha$ and $g : P_\alpha \to \mathbb{Z}_n$ sending e_1, e_2 to $0, 1_n \in \mathbb{Z}_n$ respectively. Then we prove that the following sequence:

$$1 \longrightarrow \mathbb{Z}_m \xrightarrow{f} P_\alpha \xrightarrow{g} \mathbb{Z}_n \longrightarrow 1$$

is exact. Since $f(m) = me_1 = 0$ and $0 = g(\alpha e_1) = g(ne_2) = n = 0$, we can find that f, g are well defined. Suppose that $f(k_1) = k_1 e_1 = k_2 e_1 = f(k_2)$, then $k_1 - k_2 = 0$ in $\mathbb{Z}_m$ and hence f is injective. Since $g(e_2) = 1_N$, g is a surjective mapping. And we have $g \circ f(1_M) = g(e_1) = 0$. Besides, suppose that $g(k_1 e_1 + k_2 e_2) = k_2 = 0$ in $\mathbb{Z}_n$, then we have $n|k_2$. Suppose that $k_2 = np$ and we have $k_2 e_2 = p\alpha e_1$ and hence $f(k_1 + p\alpha)k_1 e_1 + k_2 e_2$. Therefore, we have $\operatorname{Im} f = \operatorname{Ker} g$, which implies the sequence is exact. □

(5) **Proof** The sufficient and necessary condition is that $d|(\alpha - \beta)$, where $d = \gcd(m,n)$.

⇒ Suppose that we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_m \arrow[r, "f_\alpha"] \arrow[d, "\mathrm{Id}"'] & P_\alpha \arrow[r, "g_\alpha"] \arrow[d, "\phi"'] & \mathbb{Z}_n \arrow[r] \arrow[d, "\mathrm{Id}"'] & 0 \\
0 \arrow[r] & \mathbb{Z}_m \arrow[r, "f_\beta"] & P_\beta \arrow[r, "g_\beta"] & \mathbb{Z}_n \arrow[r] & 0
\end{tikzcd}$$

where ϕ is an isomorphism. And we assume that $P_\alpha =< e_1, e_2 >$ and $P_\beta =< e_1', e_2' >$ the definition in exercise 9. Then we have $g_\beta = g_\alpha \circ \phi^{-1}$ and $f_\beta^{-1} = f_\alpha^{-1} \circ \phi^{-1}$. By exercise 8, we know that $\mathbb{Z}_m \otimes \mathbb{Z}_n \simeq \mathbb{Z}_d$ and we can check that $1_m \otimes 1_n = 1_d$, where $1_m, 1_n, 1_d$ is the identity of $\mathbb{Z}_m, \mathbb{Z}_n, \mathbb{Z}_d$ respectively. Then we have:

$$\alpha 1_d = \alpha 1_m \otimes 1_n = \alpha \otimes 1_n = f_\alpha^{-1}(ne_2) \otimes g_\alpha(e_2)$$

and similarly, $\beta 1_d = f_\beta^{-1}(ne'_2) \otimes g_\beta(e'_2)$. Then we have:

$$f_\beta^{-1}(ne'_2) \otimes g_\beta(e'_2) = f_\alpha^{-1} \circ \phi^{-1}(ne'_2) \otimes g_\alpha \circ \phi^{-1}(e'_2).$$

Then we consider the difference

$$\begin{aligned} &f_\alpha^{-1}(ne_2) \otimes g_\alpha(e_2) - f_\alpha^{-1} \circ \phi^{-1}(ne'_2) \otimes g_\alpha \circ \phi^{-1}(e'_2) \\ =&f_\alpha^{-1}(n(e_2 - \phi^{-1}(e'_2))) \otimes g_\alpha(e_2) + f_\alpha^{-1} \circ \phi^{-1}(ne'_2) \otimes g_\alpha \circ \phi^{-1}(e_2 - \phi^{-1}(e'_2)) \end{aligned}$$

We can find that $g_\alpha(e_2 - \phi^{-1}(e'_2)) = 1_m - 1_m = 0$ and hence $e_2 - \phi^{-1}(e'_2) = ke_1$ for some $k \in \mathbb{Z}$ since $\operatorname{Ker} g_\alpha = \operatorname{Im} f_\alpha$. Therefore, we know that:

$$f_\alpha^{-1}(n(e_2 - \phi^{-1}(e'_2))) \otimes g_\alpha(e_2) = f_\alpha^{-1}(nke_1) \otimes g_\alpha(e_2) = nk(1_m \otimes 1_n) = 0.$$

And we can check that the second term is also 0 and hence

$$\alpha 1_d = f_\alpha^{-1}(ne_2) \otimes g_\alpha(e_2) = f_\alpha^{-1} \circ \phi^{-1}(ne'_2) \otimes g_\alpha \circ \phi^{-1}(e'_2) = \beta 1_d.$$

Therefore, we have $d|(\alpha - \beta)$.

$\Leftarrow$ Since $d|(\alpha - \beta)$, we can find a such that $m|(\alpha - \beta - an)$. Then we define $\phi(e_1) = e'_1$ and $\phi(e_2) = ae'_1 + e_2$. We can check that ϕ is well defined ($(na + \beta)e'_1 = n\phi(e_2) = \phi(ne_2) = \alpha e'_1$.) and we have:

$$f_\alpha \circ \phi = f_\beta \circ \mathrm{Id} \quad \mathrm{Id} \circ g_\alpha = g_\beta \circ \phi.$$

So we only need to prove that ϕ is bijective. By the definition, we can find that $\phi(e_1) = e'_1$ and $\phi(e_2 - ae'_1) = e'_2$ and hence we define $\psi(e'_1) = e_1$ and $\psi(e'_2) = e_2 - ae_1$. We can check that $\psi \circ \phi = \mathrm{Id}_{P_\alpha}$ and $\phi \circ \psi = \mathrm{Id}_{P_\beta}$ and hence ϕ is bijective. $\square$