MATH 6111 ALGEBRA I
PROBLEM SET 12

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1. Define $f : S^{-1}A \times M \to S^{-1}M$ by $f(a/s, m) = am/s$. Suppose $a/s = b/r$, i.e. there is $d \in S$ such that $d(ra - sb) = 0$. Then $d(ram - sbm) = 0$, which means $am/s = bm/r$. Thus, f is well-defined. We have

$$f\left(\frac{a}{s} + \frac{b}{r}, m\right) = f\left(\frac{ra+sb}{sr}, m\right) = \frac{(ra+sb)m}{sr} = \frac{ram}{sr} + \frac{sbm}{sr} = f\left(\frac{a}{s}, m\right) + f\left(\frac{b}{r}, m\right)$$

$$f\left(\frac{a}{s}, m+n\right) = \frac{a(m+n)}{s} = \frac{am}{s} + \frac{an}{s} = f\left(\frac{a}{s}, m\right) + f\left(\frac{a}{s}, n\right)$$

$$f\left(\frac{a}{s} \cdot b, m\right) = \frac{abm}{s} = f\left(\frac{a}{s}, bm\right).$$

f is a A-bilinear map, so there is $\tilde{f} : S^{-1}A \otimes_A M \to S^{-1}M$ such that $\tilde{f}(a/s \otimes m) = f(a/s, m) = am/s$. We can check that $\tilde{f}$ is a $S^{-1}A$-module homomorphism.

Define $g : S^{-1}M \to S^{-1}A \otimes_A M$ by $g(m/s) = 1/s \otimes m$. Then g is a $S^{-1}A$-module homomorphism because

$$g\left(\frac{m}{s} + \frac{n}{r}\right) = g\left(\frac{rm+sn}{sr}\right) = \frac{1}{sr} \otimes (rm+sn) = \frac{1}{sr} \otimes rm + \frac{1}{sr} \otimes sn = g\left(\frac{m}{s}\right) + g\left(\frac{n}{r}\right)$$

$$g\left(\frac{a}{t} \cdot \frac{m}{s}\right) = \frac{1}{ts} \otimes am = \frac{a}{ts} \otimes m = \frac{a}{t} \cdot g\left(\frac{m}{s}\right).$$

Note that $\tilde{f} \circ g(m/s) = \tilde{f}(1/s \otimes m) = m/s$ and $g \circ \tilde{f}(a/s \otimes m) = g(am/s) = 1/s \otimes am = a/s \otimes m$, so $\tilde{f}$ and g are inverse of each other. It follows that $S^{-1}A \otimes_A M \cong S^{-1}M$. □

5. Consider the following sequences $0 \longrightarrow M' \xrightarrow{\psi} M \xrightarrow{\phi} M'' \longrightarrow 0$ and $0 \longrightarrow M'_{\mathfrak{m}} \xrightarrow{\psi'} M_{\mathfrak{m}} \xrightarrow{\phi'} M''_{\mathfrak{m}} \longrightarrow 0$, where $\mathfrak{m}$ is a maximal ideal. $M_{\mathfrak{m}} = S^{-1}M$, where $S = A \setminus \mathfrak{m}$. For every $\mathfrak{m}$, we use ψ', ϕ' and S uniformly.

($\Rightarrow$) This is Proposition 31.2 in Lecture Notes.

($\Leftarrow$) *Lemma 1.* $(\ker\psi)_{\mathfrak{m}} = \ker\psi'$ for every maximal ideal $\mathfrak{m} \subset A$.

Proof. Suppose $m' \in \ker\psi$. Then $\psi'(m'/s) = \psi(m')/s = 0$ for any $s \in S$, i.e. $m'/s \in \ker\psi'$. This means $(\ker\psi)_{\mathfrak{m}} \subset \ker\psi'$ and this holds for any maximal ideal $\mathfrak{m} \subset A$. Conversely, suppose $m'/s \in \ker\psi'$. Then $\psi(m')/s = \psi'(m'/s) = 0$, which implies $d\psi(m') = 0$ for some $d \in S$. Thus, $m'/s \in \ker\psi'$ for any $s \in S$. Let N be the submodule generated by $\psi(m')$.

$\psi(m')/s = 0$ for any $s \in S$ implies $N_{\mathfrak{m}} = 0$. This holds for any maximal ideal $\mathfrak{m} \subset A$, so $N = 0$, i.e. $m' \in \ker\psi$. Hence, $(\ker\psi)_{\mathfrak{m}} = \ker\psi'$ for every maximal ideal $\mathfrak{m} \subset A$.

Lemma 2. $(\operatorname{im}\phi)_{\mathfrak{m}} = \operatorname{im}\phi'$ and $(M''/\operatorname{im}\phi)_{\mathfrak{m}} \cong M''_{\mathfrak{m}}/\operatorname{im}\phi'$ for every maximal ideal $\mathfrak{m} \subset A$.

Proof. By definition, $\phi'(m/s) = \phi(m)/s$, so easily $(\operatorname{im}\phi)_{\mathfrak{m}} = \operatorname{im}\phi'$.

Define $f : (M''/\operatorname{im}\phi)_{\mathfrak{m}} \to M''_{\mathfrak{m}}/\operatorname{im}\phi'$ by $f(\frac{m''+\operatorname{im}\phi}{s}) = m''/s + \operatorname{im}\phi'$. $\frac{m''+\operatorname{im}\phi}{s} = \frac{n''+\operatorname{im}\phi}{r} \Leftrightarrow d(rm'' - sn'') \in \operatorname{im}\phi$ for some $d \in S$, which implies

$$\frac{m''}{s} - \frac{n''}{r} = \frac{d(rm''-sn'')}{dsr} \in (\operatorname{im}\phi)_{\mathfrak{m}} = \operatorname{im}\phi'.$$

Conversely,

$$m''/s + \operatorname{im}\phi' = n''/r + \operatorname{im}\phi' \Leftrightarrow m''/s - n''/r \in \operatorname{im}\phi'$$
$$\Rightarrow m''/s - n''/r = a/b \text{ for some } a \in \operatorname{im}\phi \text{ and } b \in S$$
$$\Rightarrow d'(b(rm''-sn'') - sra) = 0 \text{ for some } d' \in S$$
$$\Rightarrow d'b(rm''-sn'') = d'sra \in \operatorname{im}\phi \Leftrightarrow \frac{m''+\operatorname{im}\phi}{s} = \frac{n''+\operatorname{im}\phi}{r}.$$

Thus, f is well-defined and injective. f is surjective by definition, therefore f is an isomorphism.

$0 \longrightarrow M'_{\mathfrak{m}} \xrightarrow{\psi'} M_{\mathfrak{m}} \xrightarrow{\phi'} M''_{\mathfrak{m}} \longrightarrow 0$ is an exact sequence for every maximal ideal $\mathfrak{m} \subset A$. By Lemma 1, $(\ker\psi)_{\mathfrak{m}} = \ker\psi' = 0$ for every maximal ideal $\mathfrak{m} \subset A$ and therefore $\ker\psi = 0$. Similarly, we can get $\operatorname{im}\phi = M''$ and $\ker\phi = \operatorname{im}\psi$ by applying Lemma 2 and the fact $(\ker\phi/\operatorname{im}\psi)_{\mathfrak{m}} \cong \ker\phi'/\operatorname{im}\psi'$ (which can be shown as in Lemma 2). Hence, $0 \longrightarrow M' \xrightarrow{\psi} M \xrightarrow{\phi} M'' \longrightarrow 0$ is an exact sequence. $\square$

7. Suppose J is an ideal in $A[[x]]$. Define

$$I_k = \{c \in A |\ \exists f(x) \in J \text{ such that } f(x) = cx^k + \mathit{higher}\} \cup \{0\}.$$

It is easy to check that I_k is an ideal in A and $I_0 \subset I_1 \subset \cdots$. A is Noetherian, so there is N such that $I_N = I_{N+1} = \cdots$ and each of them is finitely generated. Suppose $I_k = \langle c_{k1}, ..., c_{kn}\rangle$. Since each I_k is finite generated and there are only finite different I_k's, we can take subscript n uniformly. There is $f_{ki} \in J$ such that $f_{ki} = c_{ki}x^k + \mathit{higher}$. Let $J' = \langle f_{ki}\rangle_{0\leqslant k\leqslant N, 1\leqslant i\leqslant n}$ and claim that $J' = J$.

Each f_{ki} is in J, so $J' \subset J$. For any $f \in J$, say $f(x) = a_0 + a_1x + \cdots$. $a_0 \in I_0$, so $a_0 = b_1c_{01} + \cdots + b_nc_{0n}$ for some $b_1, \cdots, b_n \in A$. Let $g_0 = b_1f_{01} + \cdots + b_nf_{0n}$. Then $g_0 \in J' \subset J$, so $f - g_0 = a_1'x + \mathit{higher} \in J$. Continue to find $g_1, ..., g_N \in J'$ and get

$$f - g_0 - \cdots - g_N = a_{N+1}^{(N+1)}x^{N+1} + \mathit{higher},$$

where $a_{N+1}^{(N+1)} \in I_{N+1} \subset I_N$. Similarly, there exists $g_{N+1} \in \langle f_{N1}, ..., f_{Nn}\rangle \subset J'$ such that

$$f - g_0 - \cdots - g_N - xg_{N+1} = a_{N+2}^{(N+2)}x^{N+2} + \mathit{higher}.$$

Continue this process and get

$$f = g_0 + \cdots + g_N + xg_{N+1} + x^2 g_{N+2} + x^3 g_{N+3} + \cdots ,$$

where $g_{N+2}, g_{N+3}, ... \in \langle f_{N1}, ..., f_{Nn} \rangle \subset J'$. Thus, $f \in g_0 + \cdots + g_N + \langle f_{N1}, ..., f_{Nn} \rangle \subset J'$, as desired. $\square$

9. Suppose N is a submodule of $M[x]$. Define

$$L_k = \{m \in M | \ \exists f(x) \in M[x] \text{ such that } f(x) = mx^k + lower\} \cup \{0\}.$$

It is easy to check that L_k is a submodule of A and $L_0 \subset L_1 \subset \cdots$. M is a Noetherian module over A, so there is $l \in \mathbb{N}$ such that $L_l = L_{l+1} = \cdots$ and each of them is finitely generated. Suppose $L_k = \langle m_{k1}, ..., m_{kn} \rangle$. Again we take subscript n uniformly as in Problem 7. There is $f_{ki} \in N$ such that $f_{ki} = m_{ki}x^k + lower$. Let $N' = \langle f_{ki} \rangle_{0 \leqslant k \leqslant l, 1 \leqslant i \leqslant n}$ and claim that $N' = N$.

Each f_{ki} is in N, so $N' \subset N$. For any $f \in N$, define $d = \deg f$. If $d \leqslant l$, then $f \in N'$ by the choices of f_{ki}. Suppose $d > l$. Since $L_d = L_l$, there is $g_d \in \langle f_{l1}, ..., f_{ln} \rangle \subset N'$ such that $f - x^{d-l} g_d$ is of degree at most $d - 1$. Continue this process and find $g_{d-1}, ..., g_{l+1} \in \langle f_{l1}, ..., f_{ln} \rangle \subset N'$ such that $f - x^{d-l} g_d - \cdots - xg_{l+1}$ is of degree at most l. We can see that $f - x^{d-l} g_d - \cdots - xg_{l+1}$ is in N' by the case of $d \leqslant l$. Thus, $f \in N'$ as desired. We have shown that every submodule in $M[x]$ is finitely generated, therefore $M[x]$ is a Noetherian module over $A[x]$. $\square$

10. If $f(m) = 0$ for some $m \in M$, then we have $f^2(m) = f(f(m)) = f(0) = 0$ and $f^k(m) = 0$ for $k \geqslant 2$. Thus, $\ker f \subset \ker f^2 \subset \cdots$. The kernel of any module homomorphism is a submodule. Since M is a Noetherian module over A, there is $n \in \mathbb{N}$ such that $\ker f^n = \ker f^{n+1} = \cdots$.

f is surjective, so for any $m \in M$ there is $m_1 \in M$ such that $f(m_1) = m$. Then there is $m_2 \in M$ such that $f(m_2) = m_1$, i.e. $f^2(m_2) = m$. This implies f^2 is surjective. Similarly, f^k is surjective for $k \geqslant 2$.

Assume $\ker f \neq 0$. Then $f(a) = 0$ for some $a \in M - \{0\}$. f^n is surjective, so there is $m \in M$ such that $f^n(m) = a$. Then $f^{n+1}(m) = f(k) = 0$, i.e. $m \in \ker f^{n+1}$. But $\ker f^n = \ker f^{n+1}$ implies $a = f^n(m) = 0$, a contradiction. Thus, $\ker f = 0$. f is surjective, therefore f is an isomorphism. $\square$