Homework 2

Andrew Vander Werf

September 9, 2017

3. *Problem:* Let $C : G \to \mathrm{Aut}(G)$ be given by $g \mapsto C_g$ where $C_g(x) = gxg^{-1}$. Is $G \ltimes_C G$ isomorphic to $G \times G$?

   Solution: The two are isomorphic.

   Proof. The sequence
   $$\mathbf{1} \longrightarrow G \xrightarrow{\varphi} G \ltimes_C G \xrightarrow{\pi_1} G \longrightarrow \mathbf{1}$$
   for $\varphi : g \mapsto (e, g)$ is exact, since $\varphi(g_1)\varphi(g_2) = (e, e^{-1}g_1 e g_2) = (e, g_1 g_2) = \varphi(g_1 g_2)$ and $\mathrm{Im}(\varphi) = (e, G) = \mathrm{Ker}(\pi_1)$, and is split via the section $s : g \mapsto (g, e)$. Thus we only need to find a retraction $r : G \ltimes_C G \to G$. Let $r : (g, h) \mapsto gh$. By taking $g = e$, it is clear that r is surjective and satisfies $r \circ \varphi : g \mapsto g$ for all $g \in G$, so we only need to check that r is a homomorphism. Indeed,
   $$r(g_1, h_1)r(g_2, h_2) = g_1 h_1 g_2 h_2 = r(g_1 g_2, g_2^{-1} h_1 g_2 h_2) = r\big((g_1, h_1)(g_2, h_2)\big).$$
   □

6. *Problem:* Assume that there is a short exact sequence of group homomorphisms:
   $$\mathbf{1} \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} \mathbb{Z} \longrightarrow \mathbf{1}$$
   Further assume that $\mathrm{Im}(\varphi) \subset Z(B)$. Prove that this exact sequence is trivial (in particular, $B = A \times \mathbb{Z}$).

   Solution: Since ψ is surjective, there exists $b \in B$ such that $\psi(b) = 1$ and thus $\psi(b^n) = n$. Therefore we can define a section $s : n \mapsto b^n$ thus making $B = \mathbb{Z} \ltimes_\alpha A$ for some homomorphism $\alpha : \mathbb{Z} \to \mathrm{Aut}(A)$. So now, for each $\beta \in B$, we can write $\beta = s(n)\varphi(a)$ for some unique $n \in \mathbb{Z}$ and $a \in A$. So define $r : B \to A$ by $r : \beta \mapsto a$. By taking $\beta \in \mathrm{Im}(\varphi) = \mathrm{Ker}(\psi)$, it is clear that r is surjective and satisfies $r \circ \varphi : a \mapsto a$ for all $a \in A$, so we only need to check that r is a homomorphism. Indeed, since $\mathrm{Im}(\varphi) \subset Z(B)$,
   $$\begin{aligned} r(\beta_1\beta_2) &= r\big(s(n_1)\varphi(a_1)s(n_2)\varphi(a_2)\big) \\ &= r\big(s(n_1)s(n_2)\varphi(a_1)\varphi(a_2)\big) \\ &= r\big(s(n_1 n_2)\varphi(a_1 a_2)\big) \\ &= a_1 a_2 = r(\beta_1)r(\beta_2). \end{aligned}$$

7. *Problem:* Let A_1 and A_2 be two groups and G be a subgroup of $A_1 \times A_2$. Let $\pi_1 : A_1 \times A_2 \to A_1$ and $\pi_2 : A_1 \times A_2 \to A_2$ be the natural projections. Define:
   $$\begin{aligned} N_1 &= G \cap A_1; \qquad & H_1 &= \pi_1(G) \\ N_2 &= G \cap A_2; \qquad & H_2 &= \pi_2(G) \end{aligned}$$
   Prove that N_1 is normal in H_1 and N_2 is normal in H_2. Prove that there exists an isomorphism $H_1/N_1 \to H_2/N_2$.

   Solution: Let $h_1 \in H_1$ and $n \in N_1$. Then $(n, e) \in G$ and there exists $h_2 \in H_2$ such that $(h_1, h_2) \in G$. So $(h_1, h_2)(n, e)(h_1, h_2)^{-1} = (h_1 n h_1^{-1}, e) \in G$ and thus $h_1 n h_1^{-1} \in N_1$. Thus $N_1 \triangleleft H_1$ and by the same argument $N_2 \triangleleft H_2$. For every $h_1 \in H_1$, there exists an $h_2 \in H_2$ such that $(h_1, h_2) \in G$. We will show that the map $\varphi : H_1/N_1 \to H_2/N_2$ defined by $\varphi : h_1N_1 \mapsto h_2N_2$ is an isomporphism. To see that this map is well defined, suppose $(h_1, h_2), (h_1, h_2') \in G$. Then

$$(h_1^{-1}, h_2^{-1}) \in G \Longrightarrow (e, h_2^{-1}h_2') \in G \Longrightarrow h_2^{-1}h_2' \in N_2 \Longrightarrow h_2N_2 = h_2'N_2.$$

For injectivity, suppose $\varphi(h_1N_1) = \varphi(h_1'N_1) = h_2N_2$. Then

$$(h_1, h_2), (h_1', h_2) \in G \Longrightarrow (h_1^{-1}, h_2^{-1}) \in G \Longrightarrow (h_1^{-1}h_1', e) \in G \Longrightarrow h_1^{-1}h_1' \in N_1 \Longrightarrow h_1N_1 = h_1'N_1.$$

For surjectivity, any $h_2 \in H_2$ has at least one $h_1 \in H_1$ such that $(h_1, h_2) \in G$ and thus $\varphi(h_1N_1) = h_2N_2$.

8. *Problem:* Recall that $SL_2(\mathbb{C})$ is the group of 2×2 matrices of determinant 1, and $GL_2(\mathbb{C})$ is the group of invertible 2×2 matrices (with entries from the field of complex numbers). The following is the short exact sequence associated to determinant $\det : GL_2(\mathbb{C}) \to \mathbb{C}^\times = \mathbb{C}\backslash\{0\}$.

$$\mathbf{1} \longrightarrow SL_2(\mathbb{C}) \longrightarrow GL_2(\mathbb{C}) \longrightarrow \mathbb{C}^\times \longrightarrow \mathbf{1}.$$

Determine the section for this sequence and whether or not it is unique.

Solution: Define

$$\begin{aligned} s &: \mathbb{C}^\times \to GL_2(\mathbb{C}) \\ &: z \mapsto \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

It is clear that s is injective and that $\det \circ s : z \mapsto z$ for all $z \in \mathbb{C}^\times$. The section is of course not unique as we can also define

$$s : z \mapsto \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}$$

which would work just as well.

10. *Problem:* Consider the following set of elements in $A_4 \subset S_4$.

$$\{\mathrm{Id}, (12)(34), (13)(24), (14)(23)\}$$

(a) Prove that they form a normal subgroup in S_4 isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

(b) Prove that the following short exact sequence splits.

$$\mathbf{1} \longrightarrow (\mathbb{Z}/2\mathbb{Z})^2 \longrightarrow A_4 \longrightarrow A_3 \longrightarrow \mathbf{1}$$

Solution:

(a) This set contains the identity element and is made entirely out of compositions of 2-cycles so that each element is its own inverse as is the case for $(\mathbb{Z}/2\mathbb{Z})^2$. It is also easily verified that of the three nontrivial elements of this set, the composition of any two of them gives the other one as is the case for $(\mathbb{Z}/2\mathbb{Z})^2$. So any bijection between this set and $(\mathbb{Z}/2\mathbb{Z})^2$ which preserves identities will be a group isomorphism. To show this subgroup is normal, it suffices to show that it is preserved under conjugation by transpositions. Indeed, let $(a\,b)(c\,d)$ be a nontrivial element of this set. Then

$$(a\,b) \circ (a\,b)(c\,d) \circ (a\,b) = (c\,d)(a\,b) = (a\,b)(c\,d)$$

and

$$(b\,c) \circ (a\,b)(c\,d) \circ (b\,c) = (b\,d)(a\,c) = (a\,c)(b\,d)$$

both of which are elements of the set for any bijective assignment of $1, 2, 3$, and 4 to the letters a, b, c, d. Since these account for the only two distinct ways that conjugation by a transposition can occur, we are done.

(b) Let $\varphi : (\mathbb{Z}/2\mathbb{Z})^2 \to A_4$ be the injection from part (a) and let $\psi : A_4 \to A_3$ have $\mathrm{Ker}(\psi) = \mathrm{Im}(\varphi)$. Then to preserve the orders of each element, ψ must map $\{(1\,2\,3)(4), (1\,3\,2)(4)\}$ onto $\{(1\,2\,3), (1\,3\,2)\}$. Therefore for $(1, a, b) \in A_3$, one of the two maps $s : (1, a, b) \mapsto (1, a, b)(4)$ or $s : (1, a, b) \mapsto (1, b, a)(4)$ is a section for this exact sequence. Which one splits the sequence depends on whether $\psi : (1, a, b)(4) \mapsto (1, a, b)$ or $\psi : (1, a, b)(4) \mapsto (1, b, a)$.