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4)

Assume, by contradiction, that we can take $g \in N_G(L) \setminus L$. Then, $gLg^{-1} = L$ and since $P \leq N \leq L$, we have $gPg^{-1} \leq L$. If $|G| = p^r m$ where $p \nmid m$ and $|P| = p^r$, we must have $|L| = p^r m'$ where $p \nmid m' | m$. Therefore, gPg^{-1} and P are Sylow p-subgroups of L. By Sylow's theorem (part B), there exists $h \in L$ such that $P = hgPg^{-1}h^{-1}$. Therefore, $hg \in N \leq L$ and $g = h^{-1}(hg) \in L$, which is a contradiction.

✓

8)

By Sylow theorem A, we have that $n_p \equiv 1 \pmod p$ and $n_p | 2$. Since $p > 2$, then $n_p = 1$. If P is the unique Sylow p-subgroup of G, then $P \triangleleft G$ and $P \cong \mathbb{Z}_p$. By Sylow theorem A, we can find $H \leq G$ with $|H| = 2$ (i.e, a Sylow 2-subgroup). Then:

(1) $P \triangleleft G$, $H \leq G$

(2) Since $H \leq HP$ and $P \leq HP$, 2 and p divide $|HP| \leq 2p$. Therefore, $|HP| = 2p$ and $G = HP$.

(3) If $a \neq e$ and $a \in P$, then $\sigma(a) = p$. Similarly, if $a \neq e$ and $a \in H$, then $\sigma(a) = 2$. Thus $H \cap P = \{e\}$

From (1), (2) and (3), $G = H \rtimes N \cong \mathbb{Z}_2 \rtimes_\alpha \mathbb{Z}_p$ where $\alpha: \mathbb{Z}_2 \longrightarrow Aut(\mathbb{Z}_p)$ is a homomorphism.

Since $\alpha(1) \circ \alpha(1) = id = \alpha(0)$, α could be α_1 or α_2 where:

- $\alpha_1(0) = id$, $\alpha_1(1) = id$ (trivial)
- $\alpha_2(0) = id$, $\alpha_2(1)(i) = p-i$, $i=0,\dots p-1$ ($\alpha_2(1)(i)$ is the inverse of i)

Therefore, $G \cong \begin{cases} \mathbb{Z}_2 \rtimes_{\alpha_1} \mathbb{Z}_p \cong \mathbb{Z}_2 \times \mathbb{Z}_p \cong \mathbb{Z}_{2p} \text{ (cyclic)} & \gcd(2,p)=1 \\ \mathbb{Z}_2 \rtimes_{\alpha_2} \mathbb{Z}_p \end{cases}$

In $\mathbb{Z}_2 \rtimes_{\alpha_2} \mathbb{Z}_p$, if we define $s_1 = (1,0)$, $s_2 = (1, \frac{p-1}{2})$ then $s_1^2 = (1+1, \alpha_2(1)(0)+0) = (0,0)$ and

$s_2^2 = (1+1, \alpha_2(1)(\frac{p-1}{2}) + \frac{p-1}{2}) = (0, \frac{p+1}{2} + \frac{p-1}{2}) = (0,p) = (0,0)$

Moreover, $s_1 s_2 = (1,0)(1, \frac{p-1}{2}) = (1+1, \alpha_2(1)(0) + \frac{p-1}{2})$
$= (0, \frac{p-1}{2}) \Rightarrow (s_1 s_2)^p = (0, p\frac{(p-1)}{2}) = (0,0)$

In addition, $s_2 s_1 = (1, \frac{p-1}{2})(1,0) = (1+1, \alpha_2(1)(\frac{p-1}{2}) + 0)$
$= (0, \frac{p+1}{2}) \Rightarrow (s_2 s_1)^p = (0, p(\frac{p+1}{2})) = (0,0)$

then, $\mathbb{Z}_2 \rtimes_{\alpha_2} \mathbb{Z}_p = \langle s_1, s_2 : s_1^2 = s_2^2 = 1, (s_1 s_2)^p = (s_2 s_1)^p \rangle$ which is the dihedral group D_p. ✓ 10

10.1.) By Sylow theorem C, $n_q = 1, p, q$ or pq and $n_q \equiv 1 \pmod q$. Since $p<q$, we have $n_q = 1$. If Q is the only Sylow q-subgroup, then $Q \trianglelefteq G$ and G is not simple.

10.2.) In addition, if $q \not\equiv 1 \pmod p$, by Sylow theorem C we must have $n_p = 1$, and therefore the Sylow p-subgroup $P \leq G$ is normal.

Moreover,

(1) $P, Q \trianglelefteq G$

(2) Since $P \leq PQ$ and $Q \leq PQ$, then p and q divide $|PQ| \leq pq$. therefore, $G = PQ$

(3) If $a \neq e$, $a \in P \Rightarrow o(a) = p$ and if $a \neq e, a \in Q$ $\Rightarrow o(a) = q$. therefore, $P \cap Q = \{e\}$

In consequence, $G \simeq P \times Q \cong \mathbb{Z}_p \times \mathbb{Z}_q \underset{\underset{\gcd(p,q)=1}{\downarrow}}{\simeq} \mathbb{Z}_{pq}$ ✓ 10

14) If $|G| = 18 = 3^2 \cdot 2$, by Sylow theorem C, $n_3 = 1$. take $P \trianglelefteq G$ the only Sylow 3-subgroup. As we saw in class, since $|P| = 3^2$, P is either isomorphic to $\mathbb{Z}_9$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$. By Sylow theorem A, we can take $H \leq G$ with $|H| = 2$. then

(1) $H \leq G$, $P \trianglelefteq G$

(2) $H \leq HP \Rightarrow 2 \mid |HP|$ and $P \leq HP \Rightarrow 9 \mid |HP|$. therefore, $|HP| = 18 \Rightarrow HP = G$

(3) $H \cap P = \{e\}$ by order of elements of H and P.

In consequence, $G \simeq H \ltimes P \cong \mathbb{Z}_2 \ltimes P$

- Suppose $P \cong \mathbb{Z}_9$, then $G \cong \mathbb{Z}_2 \ltimes_\alpha \mathbb{Z}_9$ for $\alpha: \mathbb{Z}_2 \longrightarrow \mathrm{Aut}(\mathbb{Z}_9)$ homomorphism.

then $G \cong \begin{cases} \mathbb{Z}_2 \rtimes_\alpha \mathbb{Z}_9 \cong \mathbb{Z}_2 \times \mathbb{Z}_9 \cong \mathbb{Z}_{18} & \text{if } \alpha(0) = \alpha(1) = id \\ \mathbb{Z}_2 \rtimes_\alpha \mathbb{Z}_9 \cong D_9 & \text{if } \begin{array}{l} \alpha(0) = id \\ \alpha(i) = 9 - i \end{array} \end{cases}$ ⊕

• Suppose that $P \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. then, since $\alpha(1) \circ \alpha(1) = id$, α has four possibilities: trivial, inverse in first or second coordinate and trivial in the other, and inverse in both coordinates. then

$$G \cong \begin{cases} \mathbb{Z}_2 \rtimes_{\alpha_1} (\mathbb{Z}_3 \times \mathbb{Z}_3) \cong \mathbb{Z}_6 \times \mathbb{Z}_3 & \text{if } \alpha_1(1) = id \quad ⊕⊕ \\ \mathbb{Z}_2 \rtimes_{\alpha_2} (\mathbb{Z}_3 \times \mathbb{Z}_3) \overset{(*)}{\cong} D_3 \times \mathbb{Z}_3 & \text{if } \begin{array}{l} \alpha_2(1)(x,y) \\ = (3-x, y) \end{array} \\ \mathbb{Z}_2 \rtimes_{\alpha_3} (\mathbb{Z}_3 \times \mathbb{Z}_3) & \text{if } \alpha_3(x,y) = (3-x, 3-y) \end{cases}$$

(*) Since $D_3 \cong \mathbb{Z}_2 \rtimes_{\alpha_2'} \mathbb{Z}_3$, where $\alpha_2'(1)(x) = 3 - x$,

if $\varphi : \mathbb{Z}_2 \rtimes_{\alpha_2} (\mathbb{Z}_3 \times \mathbb{Z}_3) \to (\mathbb{Z}_2 \rtimes_{\alpha_2'} \mathbb{Z}_3) \times \mathbb{Z}_3$ is defined by $\varphi(x, (y, z)) = ((x, y), z)$,

then $\varphi((x_1, (y_1, z_1))(x_2, (y_2, z_2)))$

$= \varphi((x_1 x_2, \alpha_2(1 - x_2)(y_1, z_1) \cdot (y_2, z_2)))$

$$= \begin{cases} \text{if } x_2 = 0 \\ \varphi((x_1 x_2, ((3-y_1) y_2, z_1 z_2))) = ((x_1 x_2, y_2(3-y_1)), z_1 z_2) \\ = ((x_1 x_2, \alpha_2'(1-x_2)(y_1) y_2), z_1 z_2) = ((x_1, y_1), z_1)((x_2, y_2), z_2) \\ \text{if } x_2 = 1 \\ \varphi((x_1 x_2, (y_1 y_2, z_1 z_2))) = ((x_1, y_1), z_1)((x_2, y_2), z_2) \end{cases}$$

$\Rightarrow$ φ is homomorphism, which is clearly injective and surjective.

In consequence, $\mathbb{Z}_2 \rtimes_{\alpha_2} (\mathbb{Z}_3 \times \mathbb{Z}_3) \simeq D_3 \times \mathbb{Z}_3$
$= S_3 \times \mathbb{Z}_3$

In $\oplus$ and $\oplus\oplus$ we kot the $\underline{5}$ groups of order 18 up to isomorphism

✓ 10

17) $|G| = 5 \cdot 3 \cdot 2^2$

First, we state two facts that lead us to discard values of n_p, $p = 2, 3, 5$.

(1) Let $X = \{ P \mid P$ is Sylow p-subgroup of $G \}$ where $|G| = p^r m$. then $|X| = n_p$, and $G \circlearrowright X$ transitively by conjugation. If $P \in X$, by Stab-orbit theorem, we have

$$n_p = |G \cdot P| = \frac{|G|}{|N_G[P]|}$$

Therefore, there exist $H = N_G[P] \leq G$ with $|H| = \frac{|G|}{n_p}$, i.e, $\boxed{[G:H] = n_p}$

(2) If $H \leq G$ with $[G:H] = m$, then there exists $N \trianglelefteq G$ such that $m \mid [G:N] \mid m!$. Indeed, If $X = G/H$ is the set of left cosets and $G \circlearrowright X$ by left multiplication with an action $\varphi : G \longrightarrow Sym(X) \underset{sets}{\simeq} S_m$

then, $G/Ker\varphi \simeq Im\varphi \leq S_m \Rightarrow [G:Ker\varphi] \mid m!$

Define $N := \ker\varphi$.
Moreover, if $g \in \ker\varphi \Rightarrow g \cdot H = gH = H \Rightarrow g \in H$.
therefore, $N \leq H$. Since $[G:N] = [G:H][H:N]$
$= m\,[H:N]$
In consequence, $\boxed{m \mid [G:N] \mid m! \text{ and } N \trianglelefteq G}$

By Sylow theorem C, we have that:
$n_2 = 1, 3, 5$ or 15
$n_3 = 1, 4$ or 10
$n_5 = 1$ or 6

- Since G is simple, $\boxed{n_5 = 6.}$
- If $n_3 = 4$, by fact (1) there exists $H \leq G$ with $[G:H] = 4$ and by fact (2), there is $N \trianglelefteq G$ such that $4 \mid [G:N] \mid 4!$
Since $4 \mid [G:N] \Rightarrow N \neq G$. Since $4! = 24 < 60$, and $[G:N] \mid 4! \Rightarrow N = \{e\}$. this contradicts that G is simple. therefore, $\boxed{n_3 = 10}$
- If $n_2 = 3$, by fact (1) there exists $H \leq G$ with $[G:H] = 3$ and by fact (2), there is $N \trianglelefteq G$ such that $3 \mid [G:N] \mid 3! = 6 < 60$
then $N \neq G$ and $N \neq \{e\}$ which is a contradiction therefore, $\boxed{n_2 \neq 3}$
- Since $n_3 = 10$, there are $\underline{20}$ elements in G with order 3. Similarly, $n_5 = 6 \Rightarrow$ there are 24 elements with order 5. Therefore,

the union of all Sylow 2-subgroups should be contained in the remaining 16 elements.

- Suppose that $n_2 = 15$. Then, if $|H| = 4$, $H \leq G$, by (1) we have that $|N_G[H]| = \frac{|G|}{n_2} = \frac{60}{15} = 4$

$\Rightarrow N_G[H] = H$ for each Sylow 2-subgroup ⊕

Now, if all Sylow 2-subgroup had trivial intersection, then there would be $3 \cdot 15 = 45$ more elements of even order besides the 44 elements from Sylow 3 and 5-subgroups. However, $|G| = 60$. Therefore, we can take $H, K \leq G$ such that $|H| = |K| = 4$ and $|H \cap K| = 2$. If $H \cap K = \{e, s\}$, where $s^2 = e$, Let us prove that $|Z(s)| = 4$.

⊕⊕ If we had this, since $s \in H$, $s \in K$ and H, K are abelian, we should have $H \cup K \subseteq Z(s)$. However, we would have $|H \cup K| = 8 > 4 = |Z(s)|$, which would be a contradiction.

- More generally, let us prove that if $n_2 = 15$, and order of s is 2, $|Z(s)| = 4$. Let such $s \in G$. Since there exists K Sylow 2-subgroup in G such that $\{e, s\} \subseteq K$ (by Sylow theorem B), and K is abelian (it has order 4), then $K \leq Z(s) \leq G$.

Therefore, $4 \mid |Z(s)| \mid 60$ and in that way, $|Z(s)| = 4, 12, 20$ or 60.

- If $|Z(s)| = 60 \Rightarrow G = Z(s) \Rightarrow \forall g \in G$ $gsg^{-1} = s \Rightarrow \{e, s\} \lhd G$ which condratricts simplicity of G.
- If $|Z(s)| = 20 \Rightarrow [G : Z(s)] = 3$ and by (2) we could get $N \lhd G$ such that $3 \mid [G : N] \mid 3! = 6 \Rightarrow N \neq \{e\}$ and $N \neq G$ and contradicts simplicity again
- If $|Z(s)| = 12 = 4 \cdot 3$, and n_3' is the # of Sylow 3-subgroups of G, then $n_3' = 1$ or $n_3' = 4$
  - If $n_3' = 1 \Rightarrow \exists K \lhd Z(s)$ such that $|K| = 3$ $\Rightarrow Z(s) \leq N_G[K]$. However, since $n_3 = 10$, by (1) we have $|N_G[K]| = 60/10 = 6$ $< 12 = |Z(s)|$ (Contradiction)
  - If $n_3' = 4$, then there would be $4 \cdot 2 = 8$ non-null elements of order 3. Therefore, there is only room of $12 - 8 = 4$ elements for Sylow 2-subgroups. In consequence, there could be only one Sylow-2 subgroup of G', say $\bar{H}$. Then, $\bar{H} \lhd G'$ and in that way $G' \leq N_G[\bar{H}]$. However,

by ⊕, $4 = |N_G[H]| < |G'| = 12$
(contradiction)

- In consequence, $|Z(G)| = 4$ and by ⊕⊕ we get again another contradiction.

- therefore, $n_2 \neq 15$ and $\boxed{n_2 = 5}$

✓ 10

Note: Exercise 17 was written in the most essensial points using ideas of the professor.
Exercise 14 have some ideas taken from internet (stackexchange.com)