MATH6111 - Homework 4

Ling Zhou

September 22, 2017

Notations: In this homework, we denote the commutator group of a group G by (G, G), except for problem 6. Define

$$D^1(G) := (G, G), D^n(G) := (D^{n-1}(G), D^{n-1}(G)).$$

and

$$C^1(G) := (G, G), C^n(G) := (G, C^{n-1}(G)).$$

Problem 3.

Proof. For any $A_1 = \begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix}, A_2 = \begin{bmatrix} a_2 & b_2 \\ 0 & d_2 \end{bmatrix} \in B$, we have

$$A_1 A_2 A_1^{-1} A_2^{-1} = \frac{1}{a_1 d_1} \frac{1}{a_2 d_2} \begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ 0 & d_2 \end{bmatrix} \begin{bmatrix} d_1 & -b_1 \\ 0 & a_1 \end{bmatrix} \begin{bmatrix} d_2 & -b_2 \\ 0 & a_2 \end{bmatrix} = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}, \tag{1}$$

where $c = \frac{1}{d_1 d_2}[b_1(d_2 - a_2) + b_2(a_1 - d_1)]$. For any $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$ with $c \in \mathbb{C}$, we have

$$\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}^{-1} \tag{2}$$

is in the commutator group (B, B) of B. Therefore, $(B, B) = \left\{ \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} : c \in \mathbb{C} \right\}$.

Take two elements $C_1 = \begin{bmatrix} 1 & c_1 \\ 0 & 1 \end{bmatrix}$ and $C_2 = \begin{bmatrix} 1 & c_1 \\ 0 & 1 \end{bmatrix}$ in $D^1(B) = (B, B)$. By substituting $a_1 = a_2 = d_1 = d_2 = 1$ and $b_1 = c_1, b_2 = c_2$ into (1), we get $C_1 C_2 C_1^{-1} C_2^{-1} = I_2$, the 2×2 identity matrix. So $D^2(B) = (D^1(B), D^1(B)) = \{I_2\}$, and thus B is solvable.

We know that $C^2(B) = (B, B)$, and want to compute $C^3(B) = (B, C(B))$. For any $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in B, C = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \in C(B)$, by plugging $A_1 = A$ and $A_2 = C$ into (1), we get $ACA^{-1}C^{-1} = \begin{bmatrix} 1 & c(a-d)/d \\ 0 & 1 \end{bmatrix}$. Note that (2) also implies that any $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$ with $c \in \mathbb{C}$ is in $(B, C^2(B)) = C^3(B)$. So $C^3(B) = \left\{ \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} : c \in \mathbb{C} \right\} = C^2(B)$. In addition, $C^n(B) = (B, (\cdots, (B, C^2(B)))) = C^2(B) \neq \{I_2\}$ for any $n \geq 2$. So B is not nilpotent.

□

Problem 5.

Proof. We first claim for any $n_1 \in N_1$, $n_1 N_2 H n_1^{-1} = N_2 H$. Indeed, given any $n_2 \in N, h \in H$, we have

$$n_1 n_2 h n_1^{-1} = n_1 n_2 h n_1^{-1}(h^{-1} n_1 n_1^{-1} h) = n_1 n_2 (\underbrace{h n_1^{-1} h^{-1} n_1}_{\in (G, N_1) \subset N_2}) n_1^{-1} h.$$

Since N_2 is normal, we have $n_1[n_2(hn_1^{-1}h^{-1}n_1)]n_1^{-1} \in N_2$, and thus $n_1 n_2 h n_1^{-1} \in N_2 H$. So $n_1 N_2 H n_1^{-1} \subset N_2 H$. And then $N_2 H = n_1^{-1} n_1 N_2 H n_1^{-1} n_1 \subset n_1^{-1} N_2 H n_1 \subset N_2 H$.

Take any $n_1 \in N_1, h \in H$. Since N_2 is normal, $hN_2H = N_2hH = N_2H$. It follows that

$$n_1 h N_2 H h^{-1} n_1^{-1} = n_1 h N_2 H n_1^{-1} = n_1 N_2 h H n_1^{-1} = n_1 N_2 H n_1^{-1} = N_2 H.$$

Therefore, N_2H is normal in N_1H.

□

Problem 6.

Notation: We use $[,]$ instead of $(,)$, to denote commutators.

Proof. Suppose $|G| = p_1^{r_1} \cdots p_l^{r_l}$, where p_i's are pairwise relatively prime. By Sylow Theorems, for each i, there exists a Sylow p_i-subgroup P_i of order $p_i^{r_i}$.

(1) $\implies$ (2). If $l = 1$, then G is a p-group, and thus every Sylow subgroup of G is normal.

Suppose $l > 1$. Recall the following lemma: '*if G is nilpotent and H is a proper subgroup of G, then $H \lneq N_G(H)$, where $N_G(H)$ is the normalizer of H in G.*' Let P be any Sylow p-subgroup of G. Since G is nilpotent and P is a proper subgroup, we have $P \lneq N_G(P)$ by the lemma. If $N_G(P) \lneq G$, then $N_G(P) \lneq N_G(N_G(P))$, again by the lemma. However, Problem 4 of Set 3 gives that $N_G(N_G(P)) = N_G(P)$, which gives a contradiction. So we must have $N_(P) = G$, i.e. P is normal in G.

(2) $\implies$ (3). We prove this by induction. If $l = 1$, then G is a p-group. Suppose the statement is true for $l-1$. Let G' be the subgroup of G generated by $P_1, \cdots, P_{l-1}$. For $i = 1, \ldots, l-1$, any Sylow p_i-subgroup of G' is also a Sylow p_i-subgroup of G. Since every Sylow p_i-subgroup of G' is normal in G, it is also normal in $G' \subset G$. Because G is nilpotent, $G' < G$ is also nilpotent. By the hypothesis of induction, G' is a direct product of its Sylow p-groups. Say $G' = P_1 \times \cdots \times P_{l-1}$. Note that $G' \lhd G$.

It follows from $gcd(n, p_l) = 1$ that $G' \cap P_l = \{e\}$. And thus, $|G'P_l| = \dfrac{|G'||P_l|}{|G' \cap P_l|} = np_l^{r_l}$, which gives that $G = G'P_l$. Therefore, we get $G = P_l \rtimes G'$. Suppose P_l acts on G' by ϕ. Then for any $p, p' \in P_l, g, g' \in G'$, we have

$$pgp'g' = pp'\phi(p')(g)g' \implies p'^{-1}gp' = \phi(p')(g).$$

Note that $g^{-1}p'^{-1}gp' = g^{-1}\phi(p')(g)$ is in $G' \cap P_l$, so $\phi(p')(g) = g$ for any g and p'. Therefore, $\phi(p') = Id$ for all $p' \in P_l$, which implies that $G = G' \times P_l = P_1 \times \cdots \times P_l$.

(3) $\implies$ (1). Claim: if $G = H \times K$ is a direct product of two nilpotent groups H and K, then G is nilpotent. Then by induction, we know that the finite direct product of nilpotent groups is nilpotent. Since every p_i-group is nilpotent, $G = P_1 \times \cdots \times P_l$ is nilpotent.

Now we prove the claim. Since H and K are nilpotent, so there exist integers m, n such that $C^m(H) = \{e\}$ and $C^n(K) = \{e\}$. Without loss of generality, we suppose $m \geq n$. Let $C^i(K) = \{e\}$ for $n+1 \leq i \leq m$. Then for any i, $\forall (h,k) \in H \times K, (h_i, k_i) \in C^i(H) \times C^i(K)$,

$$\begin{aligned}[(h,k),(h_i,k_i)] &= (h,k)(h_i,k_i)(h^{-1},k^{-1})(h_i^{-1},k_i^{-1}) \\ &= (hh_ih^{-1}h_i^{-1}, kk_ik^{-1}k_i^{-1}) \\ &= ([h,h_i],[k,k_i]) \in C^{i+1}(H) \times C^{i+1}(K).\end{aligned}$$

This implies that

$$C^m(H \times K) = [H \times K, C^{m-1}(H \times K)] \subset C^m(H) \times C^m(K) = \{e\},$$

and hence $H \times K$ is nilpotent. So the claim is true.

$\square$

Problem 8.

Proof. Suppose G has a Jordan-Hölder series $\Sigma : G = G_0 \rhd G_1 \rhd \cdots \rhd G_n = \{e\}$. In other words, Σ is a strictly decreasing composition series with $G_{j+1} \rhd G_j$ for $0 \leq j \leq n-1$, and there is no strictly decreasing composition series finer than Σ.

Consider the composition series $\Sigma' : G \rhd N \rhd \{e\}$. Then there exists a common refinement Σ'' of Σ and Σ'. But Σ is a Jordan-Hölder series, so Σ'' is either the same as Σ or obtained from Σ by repeating some terms. In both cases, since N appears in Σ'', we know that N appears in Σ too. Suppose $N = G_l$ for some $0 \leq l \leq n$. Then we claim $\Sigma_N : N = G_l \rhd G_{l-1} \rhd \cdots \rhd G_n = \{e\}$ forms a Jordan-Hölder series of H. Indeed, if there is a strictly decreasing composition series finer than Σ_N, then this induces a strictly decreasing composition series of G finer than Σ.

Recall that there is a one-one correspondence

$$\{\text{the normal subgroups of } G/N\} \leftrightarrow \{\text{ normal subgroups of } G \text{ containing } N\}.$$

Therefore, $G_i/N \neq G_{i+1}/N$ iff $G_i \neq G_{i+1}$, and $G_i/N \rhd G_{i+1}/N$ iff $G_i \rhd G_{i+1}$. It follows that $\Sigma_{G/N} : N = G_0/N \rhd G_1/N \rhd \cdots \rhd G_l/N = \{N\}$ forms a Jordan-Hölder series of G/N.

Now suppose that N has a Jordan-Hölder series $\Sigma_N : N = N_0 \rhd N_1 \rhd \cdots \rhd N_m = \{e\}$, and G/N has a Jordan-Hölder series $\Sigma_{G/N} : N = G_0/N \rhd G_1/N \rhd \cdots \rhd G_l/N = \{N\}$ (here we used the one-one correspondence between the normal subgroups of G/N and normal subgroups of G containing N). Then $\Sigma : G = G_0 \rhd \cdots \rhd G_l = N = N_0 \rhd N_1 \rhd \cdots \rhd N_m = \{e\}$ forms a Jordan-Hölder series of G.

In addition, we get that $l(\Sigma) = l + m = l(\Sigma_N) + l(\Sigma_{G/N})$.

$\square$